Existence, stability and optimality for optimal control problems governed by maximal monotone operators*

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August 5, 2015

Abstract. We study optimal control problems governed by maximal monotone differential inclusions with mixed control-state constraints in infinite dimensional spaces. We obtain some existence results for this kind of dynamics and construct the discrete approximations that allows us to strongly approximate optimal solutions of the continuous-type optimal control problems by their discrete counterparts. Our approach allows us to apply our results for a wide class of mappings that are applicable in mechanics and material sciences.

Key words. optimal control, maximal monotone differential inclusions, normal cone mappings, discrete approximations, variational analysis, generalized differentiation

AMS subject classifications. 49J52, 49J53, 49K24, 49M25, 90C30

1 Introduction

Nonautonomous differential inclusions of the form

$$-\dot{x}(t) \in \mathcal{A}_t(x(t)) \quad \text{for} \quad t \in \mathcal{I}, \quad \text{and} \quad x(t_0) = \mathrm{x}_0, \tag{1.1}$$

governed by maximal monotone operators, have an important number and variety of applications in partial differential equations (heat equations and obstacle problems), mechanics (rigid-body systems with impact, Coulomb friction), electricity (diodes and transistors) and management (queueing and use of limited resources), as extensively described in [5, 7, 23, 32] and the references therein. Of particular interest is the case where, at every instant, the operator $\mathcal{A}_t$ is the subdifferential of a proper, lower-semicontinuous and convex function Φ_t,

$$-\dot{x}(t) \in \partial\Phi_t(x(t)) \quad \text{for} \quad t \in \mathcal{I}, \quad \text{and} \quad x(t_0) = \mathrm{x}_0,$$

in view of its applications in nonsmooth optimization and optimal control problems. It models, for instance, penalization or regularization procedures that yield constrained optima as time goes to $+\infty$ (see [2, 8, 13], among others). A special − yet very important − class of differential inclusions of this kind is the *sweeping process* introduced by J.-J. Moreau in the 1970s, given by

$$-\dot{x}(t) \in N(x(t); C(t)) \quad \text{for} \quad t \in \mathcal{I}, \quad \text{and} \quad x(t_0) = \mathrm{x}_0.$$

*The authors are grateful to the anonymous referee for valuable comments and suggestions, which helped improve the quality of the manuscript. This research was supported by FONDECYT grants 11140360, 3140060 & 1140829, Basal Project CMM Universidad de Chile, Millenium Nucleus ICM/FIC RC130003, Anillo Project ACT-1106, ECOS-Conicyt Project C13E03, Conicyt Redes 140183 and MathAmsud Project 15MATH-02.

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Here, $N(\cdot, C)$ is the *normal cone operator* with respect to a set C, and coincides with the subdifferential of the indicator function of C in the sense of Convex Analysis. Roughly speaking, it models the movement of a particle that is constrained to lie in a moving set, being forced to head inwards upon contact with the boundary. The sweeping process has many applications in evolutionary variational inequalities, see [7, 16, 23]. Recently, the sweeping process has gathered much attention in mathematical viewpoint, see [6, 9, 11, 22, 27, 33].

Usually, the study of the aforementioned systems relies on context-dependent techniques. In the most general setting, namely (1.1), existence and approximation results often require some geometric regularity conditions that are not applicable to the sweeping process and other relevant particular instances [1, 12, 17, 22, 33]. This paper is a contribution towards a more unified approach, especially when (1.1) describes a *controlled* dynamical system. In other words, when the time-dependence is induced by a an external action that affects the system.

Overview of our main results

Throughout this paper, H and K are real Hilbert spaces, $(t, \mathrm{x}, \mathrm{u}) \mapsto F(t, \mathrm{x}, \mathrm{u})$ is a set-valued mapping such that, for every $t \in \mathcal{I} := [0, T]$ and $\mathrm{u} \in \mathrm{K}$, $F(t, \cdot, \mathrm{u})$ is maximal monotone on H. We consider a controlled differential inclusion of the from

$$-\dot{x}(t) \in F(t, x(t), u(t)) \quad \text{a.e. } \ t \in [0, T], \qquad \text{and} \qquad x(0) = \mathrm{x}_0. \tag{1.2}$$

Here, $x : [0, T] \to \mathrm{H}$ and $u : [0, T] \to \mathrm{K}$ represent the trajectory and control of the dynamical system, respectively, and $\mathrm{x}_0 \in \mathrm{H}$ is the initial condition. The main relevant properties (for our purpose) of maximal monotone operators, along with some particular instances of (1.2), will be recalled in Section 2.

The first contribution of this research concerns several properties of the trajectories of (1.2), when an *admissible* control is given. More precisely, we obtain the following results:

- In Section 3, we prove the existence of a unique Lipschitz-continuous solution to (1.2) under assumptions that generalize [21], by replacing global estimates by local ones. In the case of sweeping process, our work improves [1, 12, 17, 22, 33] and allows us to consider unbounded moving sets. In addition, the Lipschitz constant of the solution can be estimated in several cases. A thorough comparison with the existent literature, and a detailed and commented description of the hypotheses − along with guidelines for their verification −, are given in Section 3.1. Theorem 3.8 is the main result of this section.

- In Section 4, we build a piecewise linear approximation obtained from an explicit-implicit discretization of (1.2) when $F \colon (t, \mathrm{x}, \mathrm{u}) \mapsto \partial\Phi(\mathrm{x}) + A(t, \mathrm{x}, \mathrm{u})$ and Φ is lower semicontinuous and convex, and we prove its strong convergence to a solution of (1.2) in $W^{1,2}([0, T]; \mathrm{H}) \times \mathcal{C}([0, T]; \mathrm{K})$. Our assumptions on A allows us to deal with a class of mappings which does not fit in the framework of [14]. Our results also improve those in [10]. We discuss the hypotheses and their verification, as well as the comparison with previous works in Section 4.1. The main results are presented in Section 4.2.

Our second contribution is the study of dynamic optimization problems, which have many applications in control theory, optimization, equilibrium, fixed-point theory, partial differential equations, among others. Let $\varphi : \mathrm{H} \to \overline{\mathbb{R}} := (-\infty, \infty]$, $\ell : [0,T] \times \mathrm{H} \times \mathrm{K} \times \mathrm{H} \times \mathrm{K} \to \overline{\mathbb{R}}$ and we consider the following problem:

$$\text{minimize} \quad J[x,u] := \varphi(x(T)) + \int_0^T \ell(t, x(t), u(t), \dot{x}(t), \dot{u}(t))dt, \tag{1.3}$$

subject to the constrained evolution (1.2) along with the *control-state constraint*

$$\big(x(t), u(t)\big) \in M(t) \quad \text{a.e.} \quad t \in [0,T], \tag{1.4}$$

where $M : [0,T] \rightrightarrows \mathrm{H}\times\mathrm{K}$ is a set-valued mapping. Our main results in this context are presented in Section 5, and summarized as follows:

- In Section 5.1, we prove the minimizing property of a path-following discrete approximation. Even if the problem has no solution, a minimizing sequence can be used to construct auxiliary discrete problems whose solutions inherit the minimizing property. This generalizes the results in [9, 10, 14, 15], where a solution must exist and be known. We also deal with control-state constraints and rely on less restrictive hypotheses on the operators involved.

- In Section 5.2, we prove the convergence of a direct method to a solution to the Mayer (terminal cost) problem. This result does not make use of a minimizing sequence. Even though our results do not consider state constraints, to our best knowledge, the case when the dynamic governed by (1.2) under our assumptions on F has not been studied before.

Finally, in Section 6, we provide an application to electrical circuits with ideal diodes. This provides a simple illustration, where the set $C(t)$ is unbounded and previous results cannot be applied, especially those concerning discrete approximations. We explain in detail how this example fits in our framework.

2 Preliminaries and examples

2.1 Preliminaries

Throughout this paper, H and K are real Hilbert spaces. In the case in which there is no confusion, we denote by $\langle \cdot, \cdot \rangle$ the scalar products associated to both H and K. The graph of a set-valued mapping $\mathrm{A} : \mathrm{H} \rightrightarrows \mathrm{H}$ is defined as $\operatorname{gph} \mathrm{A} := \{(\mathrm{x}, \mathrm{y}) \in \mathrm{H} \times \mathrm{H} \mid \mathrm{y} \in \mathrm{Ax}\}$. Its domain is $\operatorname{dom} \mathrm{A} = \{\mathrm{x} \in \mathrm{H} \mid \mathrm{Ax} \neq \emptyset\}$. Given $\mathrm{x} \in \operatorname{dom} \mathrm{A}$, we denote by $\|\mathrm{A}^0(\mathrm{x})\| = \inf_{\mathrm{v}\in\mathrm{Ax}} \|\mathrm{v}\|$, where $\mathrm{A}^0(\mathrm{x})$ is the element in Ax of minimal norm.

A mapping $\mathrm{A} : \mathrm{H} \rightrightarrows \mathrm{H}$ is *monotone* if

$$\langle \mathrm{y}_1 - \mathrm{y}_2, \mathrm{x}_1 - \mathrm{x}_2 \rangle \geq 0 \tag{2.5}$$

for all $(\mathrm{x}_1, \mathrm{y}_1), (\mathrm{x}_2, \mathrm{y}_2) \in \operatorname{gph} \mathrm{A}$. A monotone operator is *maximal* if its graph is not properly contained in the graph of any other monotone operator. The resolvent of a monotone operator A, denoted by J_{A}, is the single-valued nonexpansive operator $J_{\mathrm{A}} = (I + \mathrm{A})^{-1}$. It is defined for all $\mathrm{x} \in \mathrm{H}$ if, and only if, A is maximal. The *Yosida regularization* of A with index $\gamma > 0$ is

$$\mathrm{A}_\gamma = \frac{1}{\gamma}(I - J_{\gamma\mathrm{A}}). \tag{2.6}$$

Given two maximal monotone operators, A and B, defined on the same Hilbert space H, and $r > 0$, we define

$$\mathrm{dis}_r(\mathrm{A},\mathrm{B}) := \sup\left\{\frac{\langle \mathrm{y}-\bar{\mathrm{y}} \,|\, \bar{\mathrm{x}}-\mathrm{x}\rangle}{1+\|\mathrm{y}\|+\|\bar{\mathrm{y}}\|} \;\Big|\; \mathrm{x}\in \operatorname{dom}\mathrm{A}\cap r\mathbb{B}, \bar{\mathrm{x}}\in\operatorname{dom}\mathrm{B}\cap r\mathbb{B}, \mathrm{y}\in \mathrm{A}\mathrm{x}, \bar{\mathrm{y}}\in \mathrm{B}\bar{\mathrm{x}}\right\}. \quad (2.7)$$

For instance, let C and D be nonempty, closed and convex subsets of H, and set $\mathrm{A} = N(\cdot;\mathrm{C})$ and $\mathrm{B} = N(\cdot;\mathrm{D})$. Then, we deduce from [34, Lemma 3.5] that

$$\mathrm{dis}_r(\mathrm{A},\mathrm{B}) = \mathcal{H}(\mathrm{C}\cap r\mathbb{B};\mathrm{D}\cap r\mathbb{B}) \quad (2.8)$$

for all $r > 0$, where $\mathcal{H}$ stands for the Hausdorff distance. We have the following:

Lemma 2.1 *Let* A *and* B *be two maximal monotone operators on* H*, and let* $r > 0$*. The following hold:*

i) For every $\mathrm{x}\in\operatorname{dom}\mathrm{A}$, $\|\mathrm{A}_1\mathrm{x}\| \le \|\mathrm{A}^0(\mathrm{x})\|$ *and* $\|J_{\mathrm{A}}\mathrm{x}\| \le \|\mathrm{x}\| + \|\mathrm{A}^0(\mathrm{x})\|$.

ii) For every $\mathrm{x}\in\mathrm{H}$, $\|J_{\mathrm{A}}\mathrm{x}\| \le \|\mathrm{x}\| + s_{\mathrm{A}}$*, where* $s_{\mathrm{A}} = \inf_{\mathrm{y}\in\operatorname{dom}\mathrm{A}}(2\|\mathrm{y}\| + \|\mathrm{A}^0(\mathrm{y})\|)$.

iii) For every $\mathrm{x}\in\operatorname{dom}\mathrm{A}\cap r\mathbb{B}$ *and* $\gamma > 0$,

$$\|\mathrm{x} - J_{\gamma\mathrm{B}}\mathrm{x}\| \le \gamma\|\mathrm{A}^0(\mathrm{x})\| + \mathrm{dis}_{t_{\gamma\mathrm{B}}}(\mathrm{A},\mathrm{B}) + \sqrt{\gamma(1+|\mathrm{A}^0\mathrm{x}|)\mathrm{dis}_{t_{\gamma\mathrm{B}}}(\mathrm{A},\mathrm{B})}, \quad (2.9)$$

where $t_{\gamma\mathrm{B}} = r + s_{\gamma\mathrm{B}}$.

iv) For every $\mathrm{x},\bar{\mathrm{x}}\in r\mathbb{B}$,

$$\|J_{\gamma\mathrm{A}}\mathrm{x} - J_{\gamma\mathrm{B}}\bar{\mathrm{x}}\|^2 \le \|\mathrm{x}-\bar{\mathrm{x}}\|^2 + 2\gamma(1+\|\mathrm{A}_\gamma\mathrm{x}\| + \|\mathrm{B}_\gamma\bar{\mathrm{x}}\|)\mathrm{dis}_{t_\gamma}(\mathrm{A},\mathrm{B}), \quad (2.10)$$

where $t = \max\{t_{\gamma\mathrm{A}}, t_{\gamma\mathrm{B}}\} = r + \max\{s_{\gamma\mathrm{A}}, s_{\gamma\mathrm{B}}\}$.

Proof. For i), let $\mathrm{x}\in\operatorname{dom}\mathrm{A}$ and $\mathrm{u}\in\mathrm{A}\mathrm{x}$, and denote $\mathrm{p} := J_{\mathrm{A}}\mathrm{x}$. Then $(\mathrm{x}-\mathrm{p})\in\mathrm{A}\mathrm{p}$, and so

$$0 \le \langle(\mathrm{x}-\mathrm{p})-\mathrm{u}, \mathrm{p}-\mathrm{x}\rangle \le -\|\mathrm{x}-\mathrm{p}\|^2 + \langle\mathrm{u},\mathrm{x}-\mathrm{p}\rangle. \quad (2.11)$$

From the Cauchy-Schwartz inequality, we obtain $\|\mathrm{A}_1\mathrm{x}\| \le \|\mathrm{u}\|$ and the result follows by taking the infimum over $\mathrm{u}\in\mathrm{A}\mathrm{x}$. The second conclusion comes from the triangle inequality.

To prove ii), let $\mathrm{x}\in\mathrm{H}$ and $\mathrm{y}\in\operatorname{dom}\mathrm{A}$. Since J_{A} is nonexpansive, part i) gives

$$\|J_{\mathrm{A}}\mathrm{x}\| \le \|J_{\mathrm{A}}\mathrm{x} - J_{\mathrm{A}}\mathrm{y}\| + \|J_{\mathrm{A}}\mathrm{y}\| \le \|\mathrm{x}-\mathrm{y}\| + \|\mathrm{y}\| + \|\mathrm{A}^0(\mathrm{y})\| \le \|\mathrm{x}\| + 2\|\mathrm{y}\| + \|\mathrm{A}^0(\mathrm{y})\|, \quad (2.12)$$

and the result follows by taking infimum over all $\mathrm{y}\in\operatorname{dom}\mathrm{A}$.

For iii), let $\mathrm{x}\in\operatorname{dom}\mathrm{A}\cap r\mathbb{B}$. It follows from part ii) that $J_{\gamma\mathrm{B}}\mathrm{x}\in\operatorname{dom}\mathrm{B}\cap t_{\gamma\mathrm{B}}\mathbb{B}$ and, since $\mathrm{B}_\gamma\mathrm{x}\in\mathrm{B}(J_{\gamma\mathrm{B}}\mathrm{x})$ we have, for every $\mathrm{y}\in\mathrm{A}\mathrm{x}$,

$$\mathrm{dis}_{t_{\gamma\mathrm{B}}}(\mathrm{A},\mathrm{B}) \ge \frac{\langle\mathrm{y}-\mathrm{B}_\gamma\mathrm{x}, J_{\gamma\mathrm{B}}\mathrm{x}-\mathrm{x}\rangle}{1+\|\mathrm{y}\|+\|\mathrm{B}_\gamma\mathrm{x}\|} = \frac{\|\mathrm{x}-J_{\gamma\mathrm{B}}\mathrm{x}\|^2 - \gamma\langle\mathrm{y},\mathrm{x}-J_{\gamma\mathrm{B}}\mathrm{x}\rangle}{\gamma(1+\|\mathrm{y}\|)+\|\mathrm{x}-J_{\gamma\mathrm{B}}\mathrm{x}\|}, \quad (2.13)$$

where the first inequality follows from (2.7). The rest of the proof is identical to [21, Lemma 4(a)].

Finally, for iv), since $J_{\gamma\mathrm{A}}\mathrm{x}\in\operatorname{dom}\mathrm{A}\cap t_{\gamma\mathrm{A}}\mathbb{B}$, $J_{\gamma\mathrm{B}}\bar{\mathrm{x}}\in\operatorname{dom}\mathrm{B}\cap t_{\gamma\mathrm{B}}\mathbb{B}$, $\mathrm{A}_\gamma\mathrm{x}\in\mathrm{A}(J_{\gamma\mathrm{A}}\mathrm{x})$ and $\mathrm{B}_\gamma\bar{\mathrm{x}}\in\mathrm{B}(J_{\gamma\mathrm{B}}\bar{\mathrm{x}})$, from (2.7) we obtain

$$\langle\mathrm{A}_\gamma\mathrm{x}-\mathrm{B}_\gamma\bar{\mathrm{x}}, J_{\gamma\mathrm{B}}\bar{\mathrm{x}} - J_{\gamma\mathrm{A}}\mathrm{x}\rangle \le (1+\|\mathrm{A}_\gamma\mathrm{x}\|+\|\mathrm{B}_\gamma\bar{\mathrm{x}}\|)\mathrm{dis}_{t_\gamma}(\mathrm{A},\mathrm{B}), \quad (2.14)$$

and the result is obtained by following the arguments in [21, Lemma 4(b)]. △

Remark 2.2 *Note that, by taking $r \to \infty$, we recover the results in [21, Lemma 4].*

We recall that, throughout this paper, we assume that, for every $t \in [0,T]$ and $\mathrm{u} \in \mathrm{K}$, the operator $G(t,\mathrm{u}) := F(t,\cdot,\mathrm{u}) : \mathrm{H} \rightrightarrows \mathrm{H}$ is maximal monotone. For each $t \in [0,T]$, $\mathrm{u} \in \mathrm{K}$ and $\lambda > 0$, we define

$$J_\lambda^{t,\mathrm{u}} := J_{\lambda G(t,\mathrm{u})} = \big(I + \lambda G(t,\mathrm{u})\big)^{-1}. \tag{2.15}$$

For further properties and examples of monotone operators, the reader is referred to [4, 29].
We say that a function $\Phi\colon \mathrm{H} \to \mathbb{R}$ is *boundedly Lipschitz-continuous* if

$$(\forall r > 0)(\exists L_r > 0)(\forall (\mathrm{x},\mathrm{y}) \in B(0,r) \times B(0,r)) \quad |\Phi(\mathrm{x}) - \Phi(\mathrm{y})| \le L_r \|\mathrm{x} - \mathrm{y}\|. \tag{2.16}$$

In particular, any quadratic function is boundedly Lipschitz-continuous. In addition, it is easy to prove that a convex function Φ is *boundedly Lipschitz-continuous* if and only if

$$(\forall r > 0)(\exists L_r > 0)(\forall x \in B(0,r)) \quad \sup_{\mathrm{u} \in \partial\Phi(\mathrm{x})} \|\mathrm{u}\| \le L_r. \tag{2.17}$$

Given $\Delta > 0$ and $m \in I\!N$, a Δ-*regular mesh* of $[0,T]$ is a partition $\{0 = t_m^0 < \ldots < t_m^m = T\}$ such that $\Delta_m := \max\{t_m^{i+1} - t_m^i \mid i = 0,\ldots,m-1\} \le \Delta/m$. The mesh is *uniform* if $\Delta_m = t_m^{i+1} - t_m^i = T/m$ for every $i = 0,\ldots,m-1$.

2.2 Some Examples

Example 2.3 (Sweeping process). Let $\mathrm{K} := \mathrm{H} \times \mathbb{R}$ and set $\mathrm{C}(\mathrm{v},\mathrm{b}) := \{\mathrm{y} \in \mathrm{H} \mid \langle \mathrm{y},\mathrm{v}\rangle \le \mathrm{b}\}$ for each $(\mathrm{v},\mathrm{b}) \in \mathrm{K}$ with $\mathrm{v} \neq 0$. Consider the operator

$$F\big(t,\mathrm{x},\mathrm{v},\mathrm{b}\big) := N\big(\mathrm{x};\mathrm{C}(\mathrm{v},\mathrm{b})\big) = \begin{cases} \varnothing & \text{if } \langle \mathrm{v},\mathrm{x}\rangle > \mathrm{b} \\ \mathbb{R}_+\{\mathrm{v}\} := \{\alpha\mathrm{v} \mid \alpha \ge 0\} & \text{if } \langle \mathrm{v},\mathrm{x}\rangle = \mathrm{b} \\ \{0\} & \text{if } \langle \mathrm{v},\mathrm{x}\rangle < \mathrm{b}. \end{cases}$$

For every $(t,\mathrm{v},\mathrm{b}) \in [0,T] \times \mathrm{K}$, the operator $F(t,\cdot,\mathrm{v},\mathrm{b})$ is maximal monotone, and

$$J_\lambda^{t,\mathrm{v},\mathrm{b}}(\mathrm{x}) := \Big(I + \lambda F(t,\cdot,\mathrm{v},\mathrm{b})\Big)^{-1}(\mathrm{x}) = \mathrm{proj}(\mathrm{x},\mathrm{C}(\mathrm{v},\mathrm{b})). \tag{2.18}$$

Now, let $\Phi\colon \mathrm{H} \to \mathbb{R} \cup \{+\infty\}$ be a proper lower-semicontinuous convex function, and let $C\colon [0,T] \times \mathrm{U} \rightrightarrows \mathrm{H}$ be a set-valued operator with closed and convex values, where $\mathrm{U} \subset \mathrm{K}$. For each $(t,\mathrm{u},\mathrm{x}) \in [0,T] \times \mathrm{U} \times \mathrm{H}$, set

$$F(t,\mathrm{x},\mathrm{u}) = \partial\Phi(\mathrm{x}) + N(\mathrm{x},C(t,\mathrm{u})), \tag{2.19}$$

and suppose that, for every $(t,\mathrm{u}) \in [0,T] \times \mathrm{K}$, the monotone operator $F(t,\cdot,\mathrm{u})$ is maximal. This holds, for instance, if either $C(t,\mathrm{u}) \cap \mathrm{int}\,(\mathrm{dom}\,\Phi) \neq \varnothing$ or $\mathrm{int}\,(C(t,\mathrm{u})) \cap \mathrm{dom}\,\Phi \neq \varnothing$, by virtue of the Moreau-Rockafellar Theorem (see, for instance, [28, Theorem 3.30]). The class of differential inclusions (1.2) in which the maximal monotone operator $F(t,\cdot,\mathrm{u})$ is given by (2.19) includes the following important example:

Example 2.4 (Parabolic variational inequalities). Given a real Hilbert space H, consider the parabolic variational inequalities in $\mathcal{H} := L^2([0,T];\mathrm{H})$

$$\begin{cases} \Big\langle \dot{x}(t) - Ax(t) - f(t), \mathrm{x} - x(t) \Big\rangle \geq 0 \quad \forall \mathrm{x} \in \mathrm{C}(u(t)); \\ x(0) := \mathrm{x}_0; \quad x(t) \in \mathrm{C}(u(t)), \end{cases} \tag{2.20}$$

for a.e. $t \in [0,T]$, where for each $\mathrm{u} \in \mathrm{H}$, $\mathrm{C(u)} := \{\mathrm{v} \in \mathrm{H} \,|\, \mathrm{v} - \mathrm{u} \in \mathrm{K}\}$, for some nonempty closed convex cone $\mathrm{K} \subset \mathrm{H}$ and $\mathrm{A} \colon \mathrm{H} \to \mathrm{H}$ is a closed elliptic operator. This kind of problem has been extensively studied recently, see [18, 19] and their references therein. By using simple calculation, we see show that (2.20) is equivalent to

$$-\dot{x}(t) \in \mathrm{A}x(t) + f(t) + N(x(t); \mathrm{C}(u(t))) \quad \text{a.e.} \quad t \in [0,T], \quad \text{and} \quad x(0) := \mathrm{x}_0, \tag{2.21}$$

which can be seen as a particular case of (2.19) when $\Phi(x) := Ax + \langle f, x \rangle$. By considering $u : [0,T] \to \mathrm{H}$ as a control, our results allows us to study stability property as well as constructing finite approximations of (2.21) and therefore of (2.20).

Example 2.5 Consider an evolution equation

$$-\dot{x}(t) \in G(t, x(t)) \quad \text{a.e.} \quad t \in [0,T], \quad \text{and} \quad x(0) = \mathrm{x}_0, \tag{2.22}$$

where $x : [0,T] \to \mathrm{H}$ and, for every $t \in [0,T]$, $G(t,\cdot) \colon \mathrm{H} \to 2^{\mathrm{H}}$ is either maximal monotone or (one-sided) Lipschitzian. It is well known that (2.22) has a unique solution. Now given a function $u : [0,T] \to \mathrm{K}$ and a set-valued mapping $C : [0,T] \times \mathrm{K} \rightrightarrows \mathrm{H}$, consider the problem of finding a solution of (2.22) that satisfies the state constraint

$$x(t) \in C(t, u(t)) \quad \text{a.e.} \quad t \in [0,T].$$

Without further assumptions, such function may not exist. To guarantee the existence, we adjust the system (2.22) by

$$-\dot{x}(t) \in G(t, x(t)) - r(t) \quad \text{a.e.} \quad t \in [0,T], \quad \text{and} \quad x(0) = \mathrm{x}_0,$$

where the 'restoring function' $r : [0,T] \to \mathrm{H}$ is defined as follows: if $x(t)$ is in the interior of $C(t, u(t))$, we choose $r(t) := 0$. Otherwise, if $x(t)$ is on the boundary of $C(t, u(t))$, we choose $r(t) \in -N(x(t); C(t, x(t)))$ so that the trajectory $x(t)$ moves inward into and keeps staying in $C(t, u(t))$. The whole process can be described by the following system

$$-\dot{x}(t) \in G(t, x(t)) + N(x(t); C(t, x(t))) \quad \text{a.e.} \quad t \in [0,T], \quad \text{and} \quad x(0) = \mathrm{x}_0. \tag{2.23}$$

In this paper, we will study (2.23) via its existence, stability and optimality conditions.

3 Existence

In this section, we prove the existence of trajectories satisfying (1.2). Let us first introduce some notation, assumptions and preliminary results that will be useful in the sequel.

3.1 Assumptions and discussion

For every $\mathrm{u} \in \mathrm{K}$ and $t \in [0,T]$, we recall that $G(t,\mathrm{u}) := F(t,\cdot,\mathrm{u})$ is maximal monotone and that dis_r is defined in (2.7). Given $u : [0,T] \to \mathrm{K}$, we shall assume that

$(E1)$ For every $r > 0$, there exists $\alpha(u,r) > 0$ such that

$$\mathrm{dis}_r\big(G(t,u(t)), G(s,u(s))\big) \leq \alpha(u,r)(t-s) \quad \text{for all} \quad 0 \leq s \leq t \leq T. \tag{3.24}$$

$(E2)$ For every $r > 0$, there exists $\beta(u,r) > 0$ such that

$$|(G(t,u(t)))^0(\mathrm{x})| \leq \beta(u,r) \quad \text{for all} \quad (t,\mathrm{x}) \in [0,T] \times \mathrm{dom}\, G(t,u(t)) \cap r\mathbb{B}. \tag{3.25}$$

$(E3)$ There exist $\theta(u) > 0$ and $\gamma(u) > 0$ such that

$$\sup_{m\in I\!N} \max_{i=0,\ldots,m} \mathrm{dist}(0; \mathrm{dom}\, G(t_m^i, u(t_m^i))) \;\; \leq \;\; \theta(u) \tag{3.26}$$

$$\sup_{m\in I\!N} \sum_{i=1}^{m} \left\| J_{\lambda_m}^{t_m^i, u(t_m^i)}(0) - J_{\lambda_m}^{t_m^{i-1}, u(t_m^{i-1})}(0) \right\| \;\; \leq \;\; \gamma(u), \tag{3.27}$$

where, for every $m \in I\!N$, $\{0 = t_m^0 < \ldots < t_m^m = T\}$ is an uniform mesh with $\lambda_m := \Delta_m = T/m$.

We now provide some instances in which the assumption above hold. Note that, if $F\colon (t,\mathrm{x},\mathrm{u}) \mapsto \partial\Phi(\mathrm{x}) + N(\mathrm{x}; C(t,\mathrm{u}))$ and Φ is continuous, we have $\mathrm{dom}\, G(t,\mathrm{u}) = C(t,\mathrm{u})$.

Proposition 3.1 *Suppose that, for each $r > 0$, there exists $\kappa_C(r) > 0$ such that*

$$\mathcal{H}(C(t,\mathrm{u}) \cap r\mathbb{B}; C(s,\mathrm{v}) \cap r\mathbb{B}) \leq \kappa_C(r)(\|\mathrm{u}-\mathrm{v}\| + |t-s|) \tag{3.28}$$

for all $t,s \in [0,T]$ and $\mathrm{u},\mathrm{v} \in \mathrm{K}$. Assume also that $\Phi\colon \mathrm{H} \to \mathbb{R}$ is boundedly Lipschitz-continuous and $u\colon [0,T] \to \mathrm{K}$ is L_u-Lipschitz. Then, $(E1)$ holds with $F\colon (t,\mathrm{x},\mathrm{u}) \mapsto \partial\Phi(\mathrm{x}) + N(\mathrm{x}; C(t,\mathrm{u}))$.

Proof. Let $r > 0$, let $0 \leq s \leq t \leq T$, let $\mathrm{x} \in C(t,u(t)) \cap r\mathbb{B}$, let $\bar{\mathrm{x}} \in C(s,u(s)) \cap r\mathbb{B}$, let $\mathrm{y} \in \partial\Phi(\mathrm{x}) + N(\mathrm{x}; C(t,u(t)))$, and let $\bar{\mathrm{y}} \in \partial\Phi(\bar{\mathrm{x}}) + N(\bar{\mathrm{x}}; C(s,u(s)))$. Then we can choose $\mathrm{u} \in \partial\Phi(\mathrm{x})$, $\bar{\mathrm{u}} \in \partial\Phi(\bar{\mathrm{x}})$ such that $\mathrm{y}-\mathrm{u} \in N(\mathrm{x}; C(t,u(t)))$ and $\bar{\mathrm{y}}-\bar{\mathrm{u}} \in N(\bar{\mathrm{x}}; C(s,u(s)))$. Because of (2.17), $\|\mathrm{x}\| \leq r$, and $\|\bar{\mathrm{x}}\| \leq r$, there exists L_Φ^r depending only on r such that $\|\mathrm{u}\| \leq L_\Phi^r$ and $\|\bar{\mathrm{u}}\| \leq L_\Phi^r$. By using (2.8) and the fact that $\partial\Phi$ is monotone, we have

$$\begin{aligned}
\frac{\langle \mathrm{y}-\bar{\mathrm{y}} | \bar{\mathrm{x}}-\mathrm{x}\rangle}{1+\|\mathrm{y}\|+\|\bar{\mathrm{y}}\|} &= \frac{\langle (\mathrm{y}-\mathrm{u})-(\bar{\mathrm{y}}-\bar{\mathrm{u}}) | \bar{\mathrm{x}}-\mathrm{x}\rangle + \langle \mathrm{u}-\bar{\mathrm{u}} | \bar{\mathrm{x}}-\mathrm{x}\rangle}{1+\|\mathrm{y}\|+\|\bar{\mathrm{y}}\|} \\
&\leq \frac{\langle (\mathrm{y}-\mathrm{u})-(\bar{\mathrm{y}}-\bar{\mathrm{u}}) | \bar{\mathrm{x}}-\mathrm{x}\rangle}{1+\|\mathrm{y}\|+\|\bar{\mathrm{y}}\|} \\
&= \frac{\langle (\mathrm{y}-\mathrm{u})-(\bar{\mathrm{y}}-\bar{\mathrm{u}}) | \bar{\mathrm{x}}-\mathrm{x}\rangle}{1+\|\mathrm{y}-\mathrm{u}\|+\|\bar{\mathrm{y}}-\bar{\mathrm{u}}\|} \left(\frac{1+\|\mathrm{y}-\mathrm{u}\|+\|\bar{\mathrm{y}}-\bar{\mathrm{u}}\|}{1+\|\mathrm{y}\|+\|\bar{\mathrm{y}}\|}\right) \\
&\leq \mathcal{H}(C(t,u(t)) \cap r\mathbb{B}; C(s,u(s)) \cap r\mathbb{B}) \left(\frac{1+\|\mathrm{y}\|+\|\mathrm{u}\|+\|\bar{\mathrm{y}}\|+\|\bar{\mathrm{u}}\|}{1+\|\mathrm{y}\|+\|\bar{\mathrm{y}}\|}\right) \\
&\leq \kappa_C(r)(\|u(t)-u(s)\| + |t-s|)(1+2L_\Phi^r) \\
&\leq \kappa_C(r)(1+2L_\Phi^r)(L_u+1)|t-s|,
\end{aligned}$$

and $(E1)$ holds with $\alpha(u,r) := \kappa_C(r)(1+2L_\Phi^r)(L_u+1)$. △

Remark 3.2 *Note that* (3.28) *can be verified by set valued operators* $C\colon [0,T]\times \mathrm{K} \rightrightarrows \mathrm{H}$ *with unbounded values. In particular, by straightforward computation, condition* (3.28) *is satisfied when* $C(t,\mathrm{u}) = \mathrm{u}+\mathrm{C}$*, and when* $C(t,\mathrm{v},\mathrm{b}) = \{\mathrm{y}\in \mathrm{H}\,|\,\langle \mathrm{y},\mathrm{v}\rangle \le \mathrm{b}\}$ *with* $\mathrm{v}\neq 0$.

Proposition 3.3 *Assume that* $\Phi\colon \mathrm{H}\to\mathbb{R}$ *is boundedly Lipschitz-continuous. Then* ($E2$) *holds with* $F\colon (t,\mathrm{x},\mathrm{u})\mapsto \partial\Phi(\mathrm{x}) + N(\mathrm{x};C(t,\mathrm{u}))$.

Proof. For every $t\in[0,T]$ and $\mathrm{x}\in \operatorname{dom} G(t,u(t))\cap r\mathbb{B}$, we have that

$$\begin{aligned}|(G(t,u(t)))^0(\mathrm{x})| &= \inf\{\|\mathrm{y}\|\,|\,\mathrm{y}\in G(t,u(t))(\mathrm{x})\}\\ &\le \inf\{\|\mathrm{u}+\mathrm{v}\|\,|\,\mathrm{u}\in\partial\Phi(\mathrm{x});\mathrm{v}\in N(\mathrm{x};C(t,u(t)))\}\\ &\le \inf\{\|\mathrm{u}\|\,|\,\mathrm{u}\in\partial\Phi(\mathrm{x})\}.\end{aligned}$$

Because of (2.17) and $\|\mathrm{x}\|\le r$, there exists $L_\Phi^r>0$ such that $\|\mathrm{u}\|\le L_\Phi^r$ for every $\mathrm{u}\in\partial\Phi(\mathrm{x})$. Therefore, $|(G(t,u(t)))^0(\mathrm{x})|\le L_\Phi^r$ which yields ($E2$) with $\beta(u,r) := L_\Phi^r$. △

Lemma 3.4 *Let* $u\colon[0,T]\to\mathrm{K}$*, let* $m\in\mathbb{N}$*, consider a uniform mesh* $\{0=t_m^0<\cdots<t_m^m=T\}$ *and assume that* ($E2$) *and* (3.26) *hold. Then, for every* $m\in\mathbb{N}$,

$$\max_{i=0,\ldots,m}\|J_{\lambda_m}^{t_m^i,u(t_m^i)}(0)\| \le \xi(u) := 2(\theta(u)+1) + T\beta(u,\theta(u)+1), \tag{3.29}$$

where $\theta(u)$ *and* $\beta(u,r)$ *come from* ($E2$) *and* (3.26)*, respectively.*

Proof. Fix $m\in\mathbb{N}$ and fix $0\le i\le m$. From (3.26), there exists $\mathrm{y}\in\operatorname{dom} G(t_m^i,u(t_m^i))$ such that $\|\mathrm{y}\|\le\theta(u)+1$. Thus, ($E2$) and Lemma 2.1(ii), together yield

$$\|J_{\lambda_m}^{t_m^i,u(t_m^i)}(0)\|\le 2\|\mathrm{y}\| + \lambda_m|(G(t_m^i,u(t_m^i)))^0(\mathrm{y})| \le 2(\theta(u)+1)+\lambda_m\beta(u,\theta(u)+1),$$

and the result follows. △

Proposition 3.5 *Let* $\mathrm{C}\subset\mathrm{K}$ *be a nonempty closed convex set and set* $C(t,\mathrm{u}) = \mathrm{u}+\mathrm{C}$ *for* $t\in[0,T]$ *and* $\mathrm{u}\in\mathrm{K}$*. Assume that* $\Phi\colon\mathrm{H}\to\mathbb{R}$ *is boundedly Lipschitz-continuous and* $u\colon[0,T]\to\mathrm{K}$ *is of bounded variation. Then,* ($E3$) *holds with* $F\colon(t,\mathrm{x},\mathrm{u})\mapsto\partial\Phi(\mathrm{x})+N(\mathrm{x};C(t,\mathrm{u}))$.

Proof. For every $m\in\mathbb{N}$, let $\{0=t_m^0<\ldots<t_m^m=T\}$ be an uniform mesh. Since, for any $i=0,\ldots,m$, $\operatorname{dom} G(t_m^i,u(t_m^i)) = u(t_m^i)+\mathrm{C}$ we have

$$\max_{i=0,\ldots,m}\operatorname{dist}(0;\operatorname{dom} G(t_m^i,u(t_m^i))\le \max_{i=0,\ldots,m}\big(\|u(t_m^i)\|+\operatorname{dist}(0;\mathrm{C})\big)\le\theta(u), \tag{3.30}$$

where $\theta(u) := \|u(0)\| + \operatorname{dist}(0;\mathrm{C}) + \operatorname{var}(u;[0,T]) < \infty$. The last inequality follows from the bounded variation of u. Now fix $i=1,\ldots,m$ and set $\mathrm{u}_i = u(t_m^i)$ and $\mathrm{p}_i = J_{\lambda_m}^{t_m^i,\mathrm{u}_i}(0)$. Then, $-\mathrm{p}_i\in\lambda_m\partial\Phi(\mathrm{p}_i)+N(\mathrm{p}_i;\mathrm{C}+\mathrm{u}_i) = \lambda_m\partial\Phi(\mathrm{p}_i)+N(\mathrm{p}_i-\mathrm{u}_i;\mathrm{C})$. Hence, there exist $\mathrm{v}_i\in\partial\Phi(\mathrm{p}_i)$ such that $-\mathrm{p}_i-\lambda_m\mathrm{v}_i\in N(\mathrm{p}_i-\mathrm{u}_i;\mathrm{C})$. Moreover, it follows from Lemma 3.4 that $\|\mathrm{p}_i\| = \|J_{\lambda_m}^{t_m^i,\mathrm{u}_i}(0)\|\le$

$\xi(u)$, where $\xi(u)$ is defined in (3.29), and (2.17) yields $\|\mathrm{v}_i\| \leq L_\Phi^{\xi(u)}$. Therefore, the monotonicity of the mappings $N(\cdot;\mathrm{C})$ and $\partial\Phi$ yield, for every $i = 1, \ldots, m$,

$$\begin{aligned}
0 &\leq \langle -\mathrm{p}_i - \lambda_m \mathrm{v}_i + \mathrm{p}_{i-1} + \lambda_m \mathrm{v}_{i-1} | \mathrm{p}_i - \mathrm{u}_i - \mathrm{p}_{i-1} + \mathrm{u}_{i-1}\rangle \\
&= -\|\mathrm{p}_i - \mathrm{p}_{i-1}\|^2 + \langle \mathrm{p}_i - \mathrm{p}_{i-1} | \mathrm{u}_i - \mathrm{u}_{i-1}\rangle - \lambda_m \langle \mathrm{v}_i - \mathrm{v}_{i-1} | \mathrm{p}_i - \mathrm{p}_{i-1}\rangle + \lambda_m \langle \mathrm{v}_i - \mathrm{v}_{i-1} | \mathrm{u}_i - \mathrm{u}_{i-1}\rangle \\
&\leq -\|\mathrm{p}_i - \mathrm{p}_{i-1}\|^2 + \langle \mathrm{p}_i - \mathrm{p}_{i-1} | \mathrm{u}_i - \mathrm{u}_{i-1}\rangle + 2\lambda_m L_\Phi^{\xi(u)} \|\mathrm{u}_i - \mathrm{u}_{i-1}\| \\
&\leq -\|\mathrm{p}_i - \mathrm{p}_{i-1}\|^2 + \frac{1}{2}\|\mathrm{p}_i - \mathrm{p}_{i-1}\|^2 + \frac{1}{2}\|\mathrm{u}_i - \mathrm{u}_{i-1}\|^2 + 2\lambda_m L_\Phi^{\xi(u)} \|\mathrm{u}_i - \mathrm{u}_{i-1}\|
\end{aligned}$$

and, hence,

$$\|\mathrm{p}_i - \mathrm{p}_{i-1}\|^2 \leq \|\mathrm{u}_i - \mathrm{u}_{i-1}\|^2 + 4\lambda_m L_\Phi^{\xi(u)} \|\mathrm{u}_i - \mathrm{u}_{i-1}\|.$$

We deduce that, for every $m \in I\!N$ and $i \in \{1, \ldots, m\}$,

$$\left\| J_{\lambda_m}^{t_m^i, u(t_m^i)}(0) - J_{\lambda_m}^{t_m^{i-1}, u(t_m^{i-1})}(0) \right\|^2 \leq \|u(t_m^i) - u(t_m^{i-1})\|^2 + 4\lambda_m L_\Phi^{\xi(u)} \|u(t_m^i) - u(t_m^{i-1})\|$$

and, since $\lambda_m = T/m$, we deduce

$$\begin{aligned}
\sum_{i=1}^m \left\| J_{\lambda_m}^{t_m^i, u(t_m^i)}(0) - J_{\lambda_m}^{t_m^{i-1}, u(t_m^{i-1})}(0) \right\| &\leq \sum_{i=1}^m \sqrt{\|u(t_m^i) - u(t_m^{i-1})\|^2 + 4\lambda_m L_\Phi^{\xi(u)} \|u(t_m^i) - u(t_m^{i-1})\|} \\
&\leq \sum_{i=1}^m \|u(t_m^i) - u(t_m^{i-1})\| + 2\sqrt{L_\Phi^{\xi(u)}} \sum_{i=1}^m \sqrt{\lambda_m \|u(t_m^i) - u(t_m^{i-1})\|} \\
&\leq \sum_{i=1}^m \|u(t_m^i) - u(t_m^{i-1})\| + 2\sqrt{L_\Phi^{\xi(u)}} \sqrt{T \sum_{i=1}^m \|u(t_m^i) - u(t_m^{i-1})\|}.
\end{aligned}$$

Finally, since u is of bounded variation then $\sum_{i=1}^m \|u(t_m^i) - u(t_m^{i-1})\| \leq \mathrm{var}(u;[0,T])$ which implies that, for every $m \in I\!N$,

$$\sum_{i=1}^m \left\| J_{\lambda_m}^{t_m^i, u(t_m^i)}(0) - J_{\lambda_m}^{t_m^{i-1}, u(t_m^{i-1})}(0) \right\| \leq \mathrm{var}(u;[0,T]) + 2\sqrt{L_\Phi^{\xi(u)}} \sqrt{T \mathrm{var}(u;[0,T])},$$

and $(E3)$ holds with $\gamma(u) := \mathrm{var}(u;[0,T]) + 2\sqrt{L_\Phi^{\xi(u)}} \sqrt{T \mathrm{var}(u;[0,T])}$. △

Lemma 3.6 *Assume that* $(E1)$ *and* $(E2)$ *hold. Assume further that* $0 \in \mathrm{dom}\, G(t_m^i, u(t_m^i))$ *for every* $m \in I\!N$ *and* $i = 0, \ldots, m$. *Then* $(E3)$ *holds.*

Proof. It is clear that (3.26) holds with $\theta(u) = 0$. Since $0 \in \mathrm{dom}\, G(t_m^i, u(t_m^i))$, part i) of Lemma 2.1 yields $\|J_{\lambda_m}^{t_m^i, u(t_m^i)}(0)\| \leq \lambda_m |G(t_m^i, u(t_m^i))^0(0)| \leq T\beta(u,0) =: s(u)$, where the last inequality follows from $(E2)$. Then, part iv) of Lemma 2.1 and $(E1)$ together give

$$\begin{aligned}
\left\| J_{\lambda_m}^{t_m^i, u(t_m^i)}(0) - J_{\lambda_m}^{t_m^{i-1}, u(t_m^{i-1})}(0) \right\|^2 &\leq 2\lambda_m \big(1 + \|G(t_m^i, u(t_m^i))_{\lambda_m}(0)\| + \|G(t_m^{i-1}, u(t_m^{i-1}))_{\lambda_m}(0)\|\big) \\
&\qquad \times \mathrm{dis}_{s(u)}(G(t_m^i, u(t_m^i)); G(t_m^{i-1}, u(t_m^{i-1}))) \\
&\leq 2\lambda_m^2 (1 + \|G(t_m^i, u(t_m^i))_{\lambda_m}(0)\| + \|G(t_m^{i-1}, u(t_m^{i-1}))_{\lambda_m}(0)\|) \alpha(u, s(u)),
\end{aligned}$$

where $G(t,u(t))_\lambda$ is defined in (2.6). Since $0 \in \operatorname{dom} G(t_m^i, u(t_m^i))$ for every $i = 1, \ldots, m$, it follows from part i) of Lemma 2.1 and $(E2)$ that $\|G(t_m^i, u(t_m^i))_{\lambda_m}(0)\| \leq |(G(t_m^i, u(t_m^i)))^0(0)| \leq \beta(u,0)$. Therefore, we have

$$\left\| J_{\lambda_m}^{t_m^i, u(t_m^i)}(0) - J_{\lambda_m}^{t_m^{i-1}, u(t_m^{i-1})}(0) \right\| \leq \sqrt{2\alpha(u, s(u))(1+2\beta(u,0))\lambda_m}.$$

Taking the sum on i we obtain (3.27) with $\gamma(u) = T\sqrt{2\alpha(u,s)(1+2\beta(u,0))}$. △

Some of our assumptions are local versions of the corresponding ones in [21], which allows us to work on a more larger classes of problems. In particular, our framework deals with the case where $F\colon (t, \mathrm{x}, \mathrm{u}) \mapsto \partial\Phi(\mathrm{x}) + N(\mathrm{x}; C(t, \mathrm{u}))$ and the moving set $C(t, \mathrm{u})$ is not bounded, as well as Example 2.4 and Example 2.5. These assumptions allow us to make a significant improvement in comparison with recent studies on sweeping processes, see [1, 9, 10, 12, 20, 22, 27, 33], as well as studies on general maximal monotone operators, see [21, 35].

The following technical lemma will be useful in the proof of the existence result.

Lemma 3.7 *Let $u\colon [0,T] \to \mathrm{K}$ and assume that (3.26), $(E1)$ and $(E2)$ hold. Then, for every $r > 0$, $m \in I\!N$ and $i = 1, \ldots, m$, we have*

$$\sup_{\mathrm{x} \in \operatorname{dom} G(t_m^{i-1}, u(t_m^{i-1})) \cap r\mathbb{B}} \frac{1}{\lambda_m} \left\| J_{\lambda_m}^{t_m^i, u(t_m^i)}(\mathrm{x}) - \mathrm{x} \right\| \leq \sigma(u, r), \tag{3.31}$$

where $\sigma(u,r) := \beta(u,r) + \alpha(u, r+\xi(u)) + \sqrt{(1+\beta(u,r))\alpha(u, r+\xi(u))}$, and $\alpha(u,r)$, $\beta(u,r)$, and $\xi(u)$ come from $(E1)$, $(E2)$, and (3.29), respectively.

Proof. Let $r > 0$, let $\mathrm{x} \in \mathrm{H}$ with $\|\mathrm{x}\| \leq r$, fix $m \in I\!N$ and let $0 \leq i \leq m$. Lemma 3.4 and the nonexpansivity of the resolvent yield $\|J_{\lambda_m}^{t_m^i, u(t_m^i)}(\mathrm{x})\| \leq \|\mathrm{x}\| + \xi(u) \leq r + \xi(u)$, where $\xi(u)$ is defined in (3.29). Then, we deduce from Lemma 2.1(iii) that for every $\mathrm{x} \in \operatorname{dom} G(t_m^{i-1}, u(t_m^{i-1})) \cap r\mathbb{B}$, we have

$$\begin{aligned}\|J_{\lambda_m}^{t_m^i, u(t_m^i)}(\mathrm{x}) - \mathrm{x}\| \leq{}& \lambda_m |(G(t_m^{i-1}, u(t_m^{i-1})))^0(\mathrm{x})| + \operatorname{dis}_s(G(t_m^i, u(t_m^i))); G(t_m^{i-1}, u(t_m^{i-1}))) \\ &+ \sqrt{\lambda_m(1 + |(G(t_m^{i-1}, u(t_m^{i-1})))^0(\mathrm{x})|)\operatorname{dis}_s(G(t_m^i, u(t_m^i))); G(t_m^{i-1}, u(t_m^{i-1})))},\end{aligned}$$

where $s := r + \xi(u)$. From $(E2)$, $|(G(t_m^{i-1}, u(t_m^{i-1})))^0(\mathrm{x})| \leq \beta(u,r)$ and, by using $(E1)$, we obtain

$$\operatorname{dis}_s(G(t_m^i, u(t_m^i))); G(t_m^{i-1}, u(t_m^{i-1}))) \leq \alpha(u,s)\lambda_m = \alpha(u, r+\xi(u))\lambda_m.$$

As a result,

$$\|J_{\lambda_m}^{t_m^i, u(t_m^i)}(\mathrm{x}) - \mathrm{x}\| \leq \left[\beta(u,r) + \alpha(u, r+\xi(u)) + \sqrt{(1+\beta(u,r))\alpha(u, r+\xi(u))}\right]\lambda_m,$$

and (3.31) follows. △

3.2 Main result

Theorem 3.8 *Assume that* ($E1$), ($E2$) *and* ($E3$) *hold. Then, there exists a unique solution of the differential inclusion*

$$-\dot{x}(t) \in F\big(t, x(t), u(t)\big) \quad \text{a.e.} \quad t \in [0,T], \quad \textit{and} \quad x(0) = \mathrm{x}_0 \in \operatorname{dom} G(0, u(0)). \tag{3.32}$$

Moreover, x is Lipschitz-continuous with constant $\sigma(u, r(u))$, where

$$r(u) := T\sigma(u, \xi(u)) + \gamma(u) + \|\mathrm{x}_0\| + \xi(u),$$

and $\sigma(u,r)$, $\xi(u)$ and $\gamma(u)$ are defined in (3.31), (3.29) *and* ($E3$), *respectively.*

Proof. Let $m \in I\!N$ and let $\{0 = t_m^0 < \cdots < t_m^m = T\}$ be a uniform mesh with $\lambda_m := \Delta_m = T/m$. For each $i = 0, \ldots, m$, define $\mathrm{v}_m^i := J_{\lambda_m}^{t_m^i, u(t_m^i)}(0)$. It follows from Lemma 3.4 that $\max_{i=1,\ldots,m} \|\mathrm{v}_m^i\| \le \xi(u)$. Consider now the discretization algorithm

$$\mathrm{x}_m^0 := \mathrm{x}_0, \quad \text{and} \quad \mathrm{x}_m^i := J_{\lambda_m}^{t_m^i, u(t_m^i)}(\mathrm{x}_m^{i-1}) \quad \text{for} \quad i = 1, \ldots, m. \tag{3.33}$$

Note that (3.33) is equivalent to

$$\mathrm{x}_m^0 := \mathrm{x}_0, \quad \text{and} \quad -\frac{\mathrm{x}_m^i - \mathrm{x}_m^{i-1}}{\lambda_m} = \frac{1}{\lambda_m}\left(I - J_{\lambda_m}^{t_m^i, u(t_m^i)}\right)(\mathrm{x}_m^{i-1}) \quad \text{for} \quad i = 1, \ldots, m. \tag{3.34}$$

We construct the piecewise linear function x_m as follows:

$$x_m(t) := \begin{cases} \mathrm{x}_m^{i-1} + \frac{t - t_m^{i-1}}{\lambda_m}(\mathrm{x}_m^i - \mathrm{x}_m^{i-1}), & \text{if } t \in [t_m^{i-1}, t_m^i); \\ \mathrm{x}_m^m, & \text{if } t = T. \end{cases}$$

For every $i = 0, \ldots, m$, the sequence $\{\mathrm{x}_m^i\}_{m \in I\!N}$ is bounded. To verify this, we define $\{\mathrm{z}_{i,j}\}_{1 \le i \le m, 0 \le j \le i}$ as follows: $\mathrm{z}_{i,i} := \mathrm{v}_m^i$ for $1 \le i \le m$, $\mathrm{z}_{1,0} := J_{\lambda_m}^{t_m^1, u(t_m^1)}(\mathrm{x}_0)$ and for each $2 \le i \le m$, we define inductively: $\mathrm{z}_{i,j} := J_{\lambda_m}^{t_m^i, u(t_m^i)}(\mathrm{z}_{i-1,j})$ for $0 \le j \le i-1$. We can check easily from (3.33) that $\mathrm{z}_{i,0} = \mathrm{x}_m^i$ for all $i = 1, \ldots, m$. Since the mapping $J_\lambda^{t,\mathrm{u}}$ is nonexpansive, we have

$$\|\mathrm{z}_{i,j+1} - \mathrm{z}_{i,j}\| \le \left\| \mathrm{v}_m^{j+1} - J_{\lambda_m}^{t_m^{j+1}, u(t_m^{j+1})}(\mathrm{v}_m^j) \right\| \quad \text{for all} \;\; 2 \le i \le m, \; 0 \le j \le i-1,$$

which gives us the estimate

$$\begin{aligned}
\|\mathrm{v}_m^i - \mathrm{x}_m^i\| = \|\mathrm{z}_{i,i} - \mathrm{z}_{i,0}\| &\le \sum_{j=0}^{i-1} \|\mathrm{z}_{i,j+1} - \mathrm{z}_{i,j}\| \\
&\le \sum_{j=1}^{i-1} \left\| \mathrm{v}_m^{j+1} - J_{\lambda_m}^{t_m^{j+1}, u(t_m^{j+1})}(\mathrm{v}_m^j) \right\| + \|\mathrm{v}_m^1 - \mathrm{x}_m^1\| \\
&\le \sum_{j=1}^{i-1} \left\| \mathrm{v}_m^{j} - J_{\lambda_m}^{t_m^{j+1}, u(t_m^{j+1})}(\mathrm{v}_m^j) \right\| + \sum_{j=1}^{i-1} \|\mathrm{v}_m^{j+1} - \mathrm{v}_m^j\| + \|\mathrm{v}_m^1 - \mathrm{x}_m^1\| \\
&\le \sum_{j=1}^{i-1} \left\| \mathrm{v}_m^{j} - J_{\lambda_m}^{t_m^{j+1}, u(t_m^{j+1})}(\mathrm{v}_m^j) \right\| + \gamma(u) + \|J_{\lambda_m}^{t_m^1, u(t_m^1)}(0) - J_{\lambda_m}^{t_m^i, u(t_m^i)}(\mathrm{x}_0)\| \\
&\le \sum_{j=1}^{i-1} \left\| \mathrm{v}_m^{j} - J_{\lambda_m}^{t_m^{j+1}, u(t_m^{j+1})}(\mathrm{v}_m^j) \right\| + \gamma(u) + \|\mathrm{x}_0\|.
\end{aligned}$$

Now, since $\mathrm{v}_m^j = J_{\lambda_m}^{t_m^j, u(t_m^j)}(0)$, we have $\mathrm{v}_m^j \in \operatorname{dom} G(t_m^j, u(t_m^j))$. Moreover, since $\mathrm{v}_m^j \in \xi(u)\mathbb{B}$, Lemma 3.7 yields

$$\sum_{j=1}^{i-1} \|\mathrm{v}_m^j - J_{\lambda_m}^{t_m^{j+1}, u(t_m^{j+1})}(\mathrm{v}_m^j)\| \leq T\sigma(u, \xi(u))$$

for all $m \in \mathbb{N}$. As a result, $\|\mathrm{v}_m^i - \mathrm{x}_m^i\| \leq T\sigma(u,\xi(u)) + \gamma(u) + \|\mathrm{x}_0\|$, which yields $\|\mathrm{x}_m^i\| \leq T\sigma(u,\xi(u)) + \gamma(u) + \|\mathrm{x}_0\| + \xi(u) =: r(u)$. In particular,

$$\sup_{m\geq 1}\left[\max_{i=1,\ldots,m} \|\mathrm{x}_m^i\|\right] \leq r(u). \tag{3.35}$$

Therefore, $\mathrm{x}_m^{i-1} \in \operatorname{dom} G(t_m^{i-1}, u(t_m^{i-1})) \cap r(u)\mathbb{B}$ for every $i = 1, \ldots, m$, in view of (3.33). Using (3.34) and Lemma 3.7, we have

$$\left\|\frac{\mathrm{x}_m^i - \mathrm{x}_m^{i-1}}{\lambda_m}\right\| \leq \sigma(u, r(u)) \tag{3.36}$$

for every $m \geq 1$ and $i = 1, \ldots, m$. In view of the preceding discussion, we conclude that there exists a unique solution for (3.32). Notice that, unlike [21, Theorem 3], the analysis relies on the *local* assumptions $(E1)$, $(E2)$ and $(E3)$. Moreover, (3.36) leads us to the fact that the solution of (3.32) is Lipschitz-continuous with constant $\sigma(u, r(u))$. △

4 Stability and discrete approximation

In this section, we consider the case where the evolution equation is as follows:

$$-\dot{x}(t) \in \partial\Phi(x(t)) + A(t, x(t), u(t)) \quad \text{a.e.} \quad t \in [0,T], \quad x(0) = \mathrm{x}_0 \in \operatorname{dom} A(0,\cdot,u(0)), \tag{4.37}$$

with the control-state constraint

$$(x(t), u(t)) \in M(t) \quad \text{a.e.} \quad t \in [0,T], \tag{4.38}$$

where $M : [0,T] \rightrightarrows \mathrm{H} \times \mathrm{K}$, $\Phi : \mathrm{H} \to \mathbb{R}$ is a lower semicontinuous convex proper function, and $A \colon [0,T] \times \mathrm{H} \times \mathrm{K} \rightrightarrows \mathrm{H}$ need not to be maximal monotone. Now let us discuss our assumptions on A in this section.

4.1 Assumptions and discussion

$(S1)$ There exist $\delta > 0$ and $\tau > 0$ such that, for every $t \in [0,T]$, $(\bar{\mathrm{x}}, \bar{\mathrm{u}}) \in \mathrm{H} \times \mathrm{K}$, $\bar{\mathrm{v}} \in A(t, \bar{\mathrm{x}}, \bar{\mathrm{u}})$, and $\mathrm{x} \in \mathrm{H}$ such that $\|\mathrm{x} - \bar{\mathrm{x}}\| \leq \delta$, there exist $\mathrm{u} \in \mathrm{H}$ and $\mathrm{v} \in A(t, \mathrm{x}, \mathrm{u})$ such that

$$\|\mathrm{u} - \bar{\mathrm{u}}\| \leq \tau\|\mathrm{x} - \bar{\mathrm{x}}\| \ ; \ \|\mathrm{v} - \bar{\mathrm{v}}\| \leq \tau\|\mathrm{x} - \bar{\mathrm{x}}\|. \tag{4.39}$$

$(S1')$ There exists a continuous linear function $\phi : \mathrm{H} \to \mathrm{K}$, $\delta > 0$ and $\beta > 0$ such that, for every $t \in [0,T]$, $(\bar{\mathrm{x}}, \bar{\mathrm{u}}) \in \mathrm{H} \times \mathrm{K}$, $\bar{\mathrm{v}} \in A(t, \bar{\mathrm{x}}, \bar{\mathrm{u}})$, and for every $\mathrm{x} \in \mathrm{H}$ such that $\|\mathrm{x} - \bar{\mathrm{x}}\| \leq \delta$, there exist $\mathrm{u} \in \mathrm{K}$, $\mathrm{v} \in A(t, \mathrm{x}, \mathrm{u})$ satisfying

$$\mathrm{u} - \bar{\mathrm{u}} = \phi(\mathrm{x} - \bar{\mathrm{x}}) \ ; \ \|\mathrm{v} - \bar{\mathrm{v}}\| \leq \beta\|\mathrm{x} - \bar{\mathrm{x}}\|. \tag{4.40}$$

$(S2)$ There exists $\tau > 0$ such that, for any $t \in [0,T]$, $(\bar{\mathrm{x}},\bar{\mathrm{u}}) \in \mathrm{H}\times\mathrm{K}$, $\bar{\mathrm{v}} \in A(t,\bar{\mathrm{x}},\bar{\mathrm{u}})$, if $\mathrm{x} \in \mathrm{H}$ and $\eta > 0$, there exist $\mathrm{u} \in \mathrm{H}$ and $\mathrm{v} \in A(t,\mathrm{x},\mathrm{u})$ such that

$$\|\mathrm{u}-\bar{\mathrm{u}}\| \le \tau\|\mathrm{x}-\bar{\mathrm{x}}\| \ ; \ \|\mathrm{v}-\bar{\mathrm{v}}\| \le \eta. \tag{4.41}$$

Remark 4.1 *By considering $\tau = \max\{\beta, \|\phi\|\}$, it is easy to verify that $(S1')$ implies $(S1)$.*

The following results provide some instances in which the previous assumptions hold.

Proposition 4.2 *Suppose that Ψ is differentiable and that $\nabla\Psi$ is L_Ψ–Lipschitz continuous. Let $\mathrm{K} = \mathrm{H}^m \times \mathbb{R}^m$, for every $\mathrm{u} = (\mathrm{w},\mathrm{b}) \in \mathrm{U} = \{(\mathrm{w},\mathrm{b}) \in \mathrm{K} \mid \sum_{i=1}^m \|\mathrm{w}_i\|^2 = 1\}$, let $\mathrm{C}(\mathrm{u}) = \mathrm{C}(\mathrm{w},\mathrm{b}) = \{\mathrm{y} \in \mathrm{H} \mid \langle \mathrm{y},\mathrm{w}_i\rangle \le \mathrm{b}_i,\ i = 1,\dots,m\}$, and set $A\colon (t,\mathrm{x},\mathrm{u}) \mapsto \nabla\Psi(\mathrm{x}) + N(\mathrm{x};\mathrm{C}(\mathrm{u}))$. Then, $(S1)$ holds. Moreover, if $\Psi \equiv 0$, then $(S2)$ holds as well.*

Proof. Let $t \in [0,T]$, $(\bar{\mathrm{x}},\bar{\mathrm{u}}) \in \mathrm{H}\times\mathrm{K}$, $\bar{\mathrm{v}} \in A(t,\bar{\mathrm{x}},\bar{\mathrm{u}}) = \nabla\Psi(\bar{x}) + N(\bar{x};C(\bar{w},\bar{b}))$, and $\mathrm{x} \in \mathrm{H}$. We choose $\mathrm{w} = \bar{\mathrm{w}}$ and, for every $i = 1,\dots,m$, $\mathrm{b}_i = \bar{\mathrm{b}}_i + \langle \mathrm{x}-\bar{\mathrm{x}},\bar{\mathrm{w}}_i\rangle$, which guarantees that $N(\mathrm{x};\mathrm{C}(\mathrm{w},\mathrm{b})) = N(\bar{\mathrm{x}};\mathrm{C}(\bar{\mathrm{w}},\bar{\mathrm{b}}))$ and $\|\mathrm{w}-\bar{\mathrm{w}}\|^2 + \|\mathrm{b}-\bar{\mathrm{b}}\|^2 = \sum_{i=1}^m |\langle \mathrm{x}-\bar{\mathrm{x}},\bar{\mathrm{w}}_i\rangle|^2 \le \|\mathrm{x}-\bar{\mathrm{x}}\|^2 \sum_{i=1}^m \|\bar{\mathrm{w}}_i\|^2 = \|\mathrm{x}-\bar{\mathrm{x}}\|^2$. By choosing $\mathrm{v} = \bar{\mathrm{v}} - \nabla\Psi(\bar{\mathrm{x}}) + \nabla\Psi(\mathrm{x})$, $(S1)$ holds with $\tau = \max\{1, L_\Psi\}$ and any $\delta > 0$. If, additionally, $\Psi \equiv 0$, we choose $\mathrm{v} = \bar{\mathrm{v}}$ and $(S2)$ holds. △

Proposition 4.3 *Suppose that Ψ is differentiable and that $\nabla\Psi$ is L_Ψ–Lipschitz continuous. For every $\mathrm{u} \in \mathrm{K}$, let $\mathrm{C}(\mathrm{u}) = \mathrm{u} + \mathrm{C}$, where $\mathrm{C} \subset \mathrm{H}$ is nonempty, closed, and convex and set $A\colon (t,\mathrm{x},\mathrm{u}) \mapsto \nabla\Psi(\mathrm{x}) + N(\mathrm{x};\mathrm{C}(\mathrm{u}))$. Then, $(S1')$ holds. If, moreover, $\Psi \equiv 0$, $(S2)$ also holds.*

Proof. Let $t \in [0,T]$, $(\bar{\mathrm{x}},\bar{\mathrm{u}}) \in \mathrm{H}\times\mathrm{K}$, $\bar{\mathrm{v}} \in A(t,\bar{\mathrm{x}},\bar{\mathrm{u}}) = \nabla\Psi(\bar{\mathrm{x}}) + N(\bar{\mathrm{x}};\bar{\mathrm{u}}+\mathrm{C})$, and $\mathrm{x} \in \mathrm{H}$. We choose $\mathrm{u} = \bar{\mathrm{u}} - \bar{\mathrm{x}} + \mathrm{x}$, which guarantees that $N(\mathrm{x};\mathrm{C}+\mathrm{u}) = N(\bar{\mathrm{x}};\mathrm{C}+\bar{\mathrm{u}})$ and $\|\mathrm{u}-\bar{\mathrm{u}}\| = \|\mathrm{x}-\bar{\mathrm{x}}\|$. Moreover, by setting $\mathrm{v} = \bar{\mathrm{v}} - \nabla\Psi(\bar{\mathrm{x}}) + \nabla\Psi(\mathrm{x})$, we have $\mathrm{v} \in \nabla\Psi(\mathrm{x}) + N(\bar{\mathrm{x}};\mathrm{C}+\bar{\mathrm{u}}) = \nabla\Psi(\mathrm{x}) + N(\mathrm{x};\mathrm{C}+\mathrm{u})$ and $\|\mathrm{v}-\bar{\mathrm{v}}\| = \|\nabla\Psi(\mathrm{x}) - \nabla\Psi(\bar{\mathrm{x}})\| \le L_\Psi\|\mathrm{x}-\bar{\mathrm{x}}\|$. Hence $(S1')$ holds with $\beta = L_\Psi$, $\phi = I$, and any $\delta > 0$. As before, if $\Psi \equiv 0$, we choose $\mathrm{v} = \bar{\mathrm{v}}$ and $(S2)$ holds. △

Proposition 4.3 allows us to apply our result, in particular, in Example 2.4 and Example 2.5. This mapping does not possess neither the growth condition nor the Kamke continuity property introduced in [15]. Therefore, the results in [14, 15] cannot deal with our setting. The results in [9, 10] tackle the particular case of (4.37) when $\Phi \equiv 0$ and $A(t,\mathrm{x},\mathrm{u}) = N(\mathrm{x},\mathrm{C}(\mathrm{u}))$, where $\mathrm{C}(\mathrm{u})$ is defined in Proposition 4.2. Clearly, Example 2.4 and Example 2.5 are not the case.

4.2 Our results

Our first result in this section is the convergence of a *forward-backward* discrete approximation to a solution $(\bar{x},\bar{u})$ to (4.37) by assuming the existence of that solution in $\mathcal{C}^1([0,T];\mathrm{H})\times\mathcal{C}([0,T];\mathrm{K})$. This regularity property is weakened in Theorem 4.8.

Theorem 4.4 *Let $(\bar{x},\bar{u}) \in \mathcal{C}^1([0,T];\mathrm{H})\times\mathcal{C}([0,T];\mathrm{K})$ be a feasible solution to (4.37) and assume that A satisfies $(S1)$. Suppose that*

$$\begin{cases} \text{there exists } \bar{f} \in \mathcal{C}([0,T];\mathrm{H}) \text{ such that} \\ (\forall t \in [0,T]) \quad \bar{f}(t) \in A(t,\bar{x}(t),\bar{u}(t)) \quad \text{and} \quad -\dot{\bar{x}}(t) \in \partial\Phi(\bar{x}(t)) + \bar{f}(t). \end{cases} \tag{4.42}$$

Then, for every $m \in I\!N$ and $\Delta > 0$, and for any Δ-regular mesh $\{0 = t_m^0 < \ldots < t_m^m = T\}$, there exist piecewise linear functions x_m and u_m such that $(x_m(0), u_m(0)) := (\mathrm{x}_0, \bar{u}(0))$, $\big(x_m, u_m\big)$ strongly converges to $\big(\bar{x}, \bar{u}\big)$ in $W^{1,2}\big([0,T]; \mathrm{H}\big) \times \mathcal{C}\big([0,T]; \mathrm{K}\big)$, and

$$-\frac{x_m(t_m^{i+1}) - x_m(t_m^i)}{t_m^{i+1} - t_m^i} \in \partial\Phi(x_m(t_m^{i+1})) + A\big(t_m^i, x_m(t_m^i), u_m(t_m^i)\big),\ i = 0, \ldots, m-1. \tag{4.43}$$

Proof. Let $\delta > 0$ and $\tau > 0$ be given by $(S1)$, set $0 < \epsilon < \frac{\tau\delta}{e^{\tau\Delta}-1}$. For every $m \in I\!N$ we define $\eta_m^0 = 0$ and, for every $i = 0, \ldots, m-1$,

$$\begin{cases} \mathcal{E}_m^i := \|\bar{x}(t_m^i) - \bar{x}(t_m^{i+1}) + (t_m^{i+1} - t_m^i)\dot{\bar{x}}(t_m^{i+1})\| + (t_m^{i+1} - t_m^i)\|\bar{f}(t_m^{i+1}) - \bar{f}(t_m^i)\| \\ \eta_m^{i+1} := (1 + \tau(t_m^{i+1} - t_m^i))\eta_m^i + \mathcal{E}_m^i. \end{cases} \tag{4.44}$$

Using the fact that $\bar{x} \in \mathcal{C}^1$ and $\bar{f} \in \mathcal{C}$, there exists $m_0 \in I\!N$ such that $\mathcal{E}_m^i < (t_m^{i+1} - t_m^i)\epsilon$ for all $m \geq m_0$ and $i = 0, \ldots, m-1$. We can make the upper estimate for the sequence $\{\eta_m^i\}_{i=\overline{0,m}}$ as follows:

$$\eta_m^i \leq \frac{(1+\tau\Delta_m)^i - 1}{(1+\tau\Delta_m) - 1}\epsilon\Delta_m = \frac{\epsilon}{\tau}\Big[(1+\tau\Delta_m)^i - 1\Big] \leq \frac{\epsilon}{\tau}\Big[\Big(1+\frac{\tau\Delta}{m}\Big)^m - 1\Big] \leq \frac{\epsilon}{\tau}\big[e^{\tau\Delta} - 1\big] < \delta, \tag{4.45}$$

where the first inequality is easily obtained by induction. Observe that (4.45) implies

$$\lim_{m\to\infty}\Big[\sup_{0\leq i\leq m} \eta_m^i\Big] = 0. \tag{4.46}$$

We continue with the construction of $(x_m(t), u_m(t))$, also by induction. First define $\mathrm{x}_m^0 := \mathrm{x}_0$ and $\mathrm{u}_m^0 := \bar{u}(0)$. Suppose that the value of x_m^i is known and $\|\mathrm{x}_m^i - \bar{x}(t_m^i)\| \leq \eta_m^i \leq \delta$. By using $(S1)$ and the fact that $\bar{f}(t_m^i) \in A(t_m^i, \bar{x}(t_m^i), \bar{u}(t_m^i))$, one may choose u_m^i and $\mathrm{f}_m^i \in A\big(t_m^i, \mathrm{x}_m^i, \mathrm{u}_m^i\big)$ such that

$$\|\mathrm{u}_m^i - \bar{u}(t_m^i)\| \leq \tau\|\mathrm{x}_m^i - \bar{x}(t_m^i)\|, \tag{4.47}$$

and

$$\|\mathrm{f}_m^i - \bar{f}(t_m^i)\| \leq \tau\|\mathrm{x}_m^i - \bar{x}(t_m^i)\|. \tag{4.48}$$

We define

$$\mathrm{x}_m^{i+1} := \big(I + (t_m^{i+1} - t_m^i)\partial\Phi\big)^{-1}(\mathrm{x}_m^i - (t_m^{i+1} - t_m^i)\mathrm{f}_m^i) \tag{4.49}$$

or, equivalently,

$$-\frac{\mathrm{x}_m^{i+1} - \mathrm{x}_i^m}{t_m^{i+1} - t_m^i} \in \partial\Phi(\mathrm{x}_m^{i+1}) + \mathrm{f}_m^i,\ i = 0, \ldots, m-1.$$

Therefore, by constructing the piecewise linear functions $x_m(\cdot)$ satisfying $x_m(t_m^i) = \mathrm{x}_m^i$ for all $i = 1, \ldots, m$, (4.43) holds. We now construct in a similar way the piecewise linear function $u_m(\cdot)$ and we generate a piecewise constant function $f_m(\cdot)$ such that, for every $i = 1, \ldots, m$, it is equal to f_m^i on the subinterval $[t_m^i, t_m^{i+1})$ and equal to f_m^m at T. On the other hand, applying (4.42) with $t = t_m^{i+1}$, we have

$$-\dot{\bar{x}}(t_m^{i+1}) \in \partial\Phi(\bar{x}(t_m^{i+1}) + \bar{f}(t_m^{i+1}),$$

or

$$\bar{x}(t_m^{i+1}) := \big(I + (t_m^{i+1} - t_m^i)\partial\Phi\big)^{-1}\Big(\bar{x}(t_m^{i+1}) - (t_m^{i+1} - t_m^i)\dot{\bar{x}}(t_m^{i+1}) - (t_m^{i+1} - t_m^i)\bar{f}(t_m^{i+1})\Big).$$

Combining this and (4.49), we come up to

$$\begin{aligned}\|\mathrm{x}_m^{i+1}-\bar{x}(t_m^{i+1})\| \leq & \|\mathrm{x}_m^i-(t_m^{i+1}-t_m^i)\mathrm{f}_m^i-\bar{x}(t_m^{i+1})+(t_m^{i+1}-t_m^i)\dot{\bar{x}}(t_m^{i+1})+(t_m^{i+1}-t_m^i)\bar{f}(t_m^{i+1})\| \\ \leq & \|\mathrm{x}_m^i-\bar{x}(t_m^i)\|+(t_m^{i+1}-t_m^i)\|\mathrm{f}_m^i-\bar{f}(t_m^i)\|+\mathcal{E}_m^i \\ \leq & \big(1+\tau(t_m^{i+1}-t_m^i)\big)\|\mathrm{x}_m^i-\bar{x}(t_m^i)\|+\mathcal{E}_m^i \leq \eta_m^{i+1}.\end{aligned}$$

and we can repeat the whole step above to construct the whole sequence $\{\mathrm{x}_m^i,\mathrm{u}_m^i\}_{i=0,\ldots,m}$ such that $\|\mathrm{x}_m^i-\bar{x}(t_m^i)\| \leq \eta_m^i$ for all $i=1,\ldots,m$. Using (4.48), we have $\|\mathrm{f}_m^i-\bar{f}(t_m^i)\| \leq \tau\eta_m^i$ and, hence, the sequence of functions f_m converge strongly to $\bar{f}$ in $L^2([0,T];\mathrm{H})$, in view of (4.46). By using [35, Theorem 2], we obtain the convergence of x_m to $\bar{x}$ in $W^{1,2}([0,T];\mathrm{H})$ and (4.47) yields the convergence of u_m to $\bar{u}$ in $C([0,T];\mathrm{K})$. △

Remark 4.5 *If Φ is differentiable, condition (4.42) automatically holds.*

Remark 4.6 *Suppose that, in addition, $(\bar{x},\bar{u})$ satisfies (4.38). Then, from the proof of Theorem 4.4, we deduce that,*

$$(\mathrm{x}_m^i,\mathrm{u}_m^i)\in M_{\xi\eta_m^i}(t_m^i) \quad \forall i=0,\ldots,m,$$

where $\xi=\max\{1,\tau\}$ and

$$M_\epsilon(t):=\{(\mathrm{x},\mathrm{u})\in\mathrm{H}\times\mathrm{K}\mid\|\mathrm{x}-\bar{\mathrm{x}}\|\leq\epsilon,\|\mathrm{u}-\bar{\mathrm{u}}\|\leq\epsilon \text{ for some } (\bar{\mathrm{x}},\bar{\mathrm{u}})\in M(t)\}. \tag{4.50}$$

It is worth mentioning that η_m^i only depends on $\bar{x},\bar{f},\tau,t_m^0,\ldots,t_m^i$.

Proposition 4.7 *In addition to the assumptions of Theorem 4.4, suppose that A satisfies $(S1')$ and $\bar{u}\in\mathcal{C}^1([0,T];\mathrm{K})$. Then, the sequence of piecewise linear functions $(x_m,u_m)_{m\in\mathbb{N}}$ obtained from Theorem 4.4 satisfy, additionally, that $u_m(\cdot)$ converges to $\bar{u}(\cdot)$ in $W^{1,2}\big([0,T];\mathrm{H}\big)$.*

Proof. Firstly, it follows from $\big(\bar{x},\bar{u}\big)\in\mathcal{C}^1([0,T];\mathrm{H})\times\mathcal{C}^1([0,T];\mathrm{K})$ that

$$\begin{cases}\sigma_m^x:=\sum_{i=0}^{m-1}\int_{t_m^i}^{t_m^{i+1}}\left\|\frac{\bar{x}(t_m^{i+1})-\bar{x}(t_m^i)}{t_m^{i+1}-t_m^i}-\dot{\bar{x}}(t)\right\|^2dt\to 0\\ \sigma_m^u:=\sum_{i=0}^{m-1}\int_{t_m^i}^{t_m^{i+1}}\left\|\frac{\bar{u}(t_m^{i+1})-\bar{u}(t_m^i)}{t_m^{i+1}-t_m^i}-\dot{\bar{u}}(t)\right\|^2dt\to 0.\end{cases} \tag{4.51}$$

Moreover, since $(S1')$ implies $(S1)$ (see Remark 4.1), Theorem 4.4 provides a sequence of piecewise linear functions $(x_m,u_m)_{m\in\mathbb{N}}$ satisfying (4.43) and which converges strongly to $\big(\bar{x},\bar{u}\big)$ in $W^{1,2}\big([0,T];\mathrm{H}\big)\times\mathcal{C}\big([0,T];\mathrm{K}\big)$. Hence, the linearity of ϕ obtained from $(S1')$ and (4.51) yield

$$\begin{aligned}&\int_0^T\|\dot{u}_m(t)-\dot{\bar{u}}(t)\|^2dt=\sum_{i=0}^{m-1}\int_{t_m^i}^{t_m^{i+1}}\left\|\frac{u_m(t_m^{i+1})-u_m(t_m^i)}{t_m^{i+1}-t_m^i}-\dot{\bar{u}}(t)\right\|^2dt\\ &=\sum_{i=0}^{m-1}\int_{t_m^i}^{t_m^{i+1}}\left\|\frac{u_m(t_m^{i+1})-u_m(t_m^i)}{t_m^{i+1}-t_m^i}-\frac{\bar{u}(t_m^{i+1})-\bar{u}(t_m^i)}{t_m^{i+1}-t_m^i}+\frac{\bar{u}(t_m^{i+1})-\bar{u}(t_m^i)}{t_m^{i+1}-t_m^i}-\dot{\bar{u}}(t)\right\|^2dt\\ &\leq 2\sum_{i=0}^{m-1}\int_{t_m^i}^{t_m^{i+1}}\left\|\phi\Big(\frac{x_m(t_m^{i+1})-x_m(t_m^i)}{t_m^{i+1}-t_m^i}-\frac{\bar{x}(t_m^{i+1})-\bar{x}(t_m^i)}{t_m^{i+1}-t_m^i}\Big)\right\|^2dt+2\sigma_m^u\\ &\leq 2\|\phi\|^2\sum_{i=0}^{m-1}\int_{t_m^i}^{t_m^{i+1}}\left\|\frac{x_m(t_m^{i+1})-x_m(t_m^i)}{t_m^{i+1}-t_m^i}-\frac{\bar{x}(t_m^{i+1})-\bar{x}(t_m^i)}{t_m^{i+1}-t_m^i}\right\|^2dt+2\sigma_m^u\\ &\leq 2\|\phi\|^2\sum_{i=0}^{m-1}\int_{t_m^i}^{t_m^{i+1}}\left\|\dot{x}_m(t)-\dot{\bar{x}}(t)\right\|^2dt+2\|\phi\|^2\sigma_m^x+2\sigma_m^u\longrightarrow 0\end{aligned}$$

as $m \to \infty$. Hence, the result follows from $u_m(0) = \bar{u}(0)$. $\triangle$

Note that the assumption $\bar{x} \in \mathcal{C}^1$, although strictly necessary in Theorem 4.4, may be restrictive in applications. For example, in the case $\Phi(\mathrm{x}) = |\mathrm{x}|$ for $\mathrm{x} \in \mathbb{R}$, the absolutely continuous solution of the evolution equation $-\dot{x}(t) \in 1 + \partial\Phi(x(t)), x(0) = \mathrm{x}_0 > 0$ has the form

$$\begin{cases} \bar{x}(t) = -2t + \mathrm{x}_0, & \text{if} \quad t \leq \mathrm{x}_0/2; \\ \bar{x}(t) = 0, & \text{if} \quad t > \mathrm{x}_0/2. \end{cases}$$

which does not belong to $\mathcal{C}^1$. In the next result, we show that by assuming the stronger property $(S2)$ on A, we can weaken the smoothness condition on $\bar{x}$.

Theorem 4.8 *Let $\big(\bar{x}(\cdot), \bar{u}(\cdot)\big) \in W^{1,2}\big([0,T];\mathrm{H}\big) \times \mathcal{C}([0,T];\mathrm{K})$ be a feasible solution of* (4.37) *and assume that A satisfies $(S2)$. Suppose that*

$$\begin{cases} \text{there exists } \bar{f} \in L^2([0,T];\mathrm{H}) \text{ such that} \\ \text{for almost all } t \in [0,T], \quad \bar{f}(t) \in A(t, \bar{x}(t), \bar{u}(t)) \quad \text{and} \quad -\dot{\bar{x}}(t) \in \partial\Phi(\bar{x}(t)) + \bar{f}(t). \end{cases} \tag{4.52}$$

Then for any mesh $\{0 = t_m^0 < \ldots < t_m^m = T\}$ such that the inclusions on (4.52) *hold at $t_m^0, \ldots, t_m^m$ and such that $\Delta_m \to 0$ as $m \to \infty$, there exists a sequence of piecewise linear functions (x_m, u_m) satisfying* (4.43) *and $(x_m(0), u_m(0)) := (\mathrm{x}_0, \bar{u}(0))$, such that (x_m, u_m) strongly converges to $(\bar{x}, \bar{u})$ in $W^{1,2}\big([0,T];\mathrm{H}\big) \times C\big([0,T];\mathrm{K}\big)$.*

Proof. Similar to Theorem 4.4, we define $\mathrm{x}_m^0 := x_0$ and $\mathrm{u}_m^0 := \bar{u}(0)$ and suppose that the value of x_m^i is known. By using $(S2)$ and the fact that $\bar{f}(t_m^i) \in A(t_m^i, \bar{x}(t_m^i), \bar{u}(t_m^i))$, one may choose u_m^i and $\mathrm{f}_m^i \in A\big(t_m^i, \mathrm{x}_m^i, \mathrm{u}_m^i\big)$ such that

$$\|\mathrm{u}_m^i - \bar{u}(t_m^i)\| \leq \tau \|\mathrm{x}_m^i - \bar{x}(t_m^i)\|, \tag{4.53}$$

and

$$\|\mathrm{f}_m^i - \bar{f}(t_m^i)\| \leq \frac{1}{m}.$$

Hence, as in Theorem 4.4, we can guarantee the convergence of the piecewise constant function f_m to $\bar{f}$ in $L^2([0,T];\mathrm{H})$. By using [35, Theorem 2], we obtain the convergence of x_m to $\bar{x}$ in $W^{1,2}([0,T];\mathrm{H})$. Again, by using (4.53), we obtain the convergence of u_m to $\bar{u}$ in $\mathcal{C}([0,T];\mathrm{K})$. $\triangle$

Remark 4.9 *With Theorem 4.8, we can deal with nondifferentiable solutions. Under differentiability, the method in Theorem 4.4 allows us to estimate the errors $\|x_m(t_m^i) - \bar{x}(t_m^i)\|$ and $\|u_m(t_m^i) - \bar{u}(t_m^i)\|$ explicitly in terms of the original data $\bar{x}, \bar{f}, \tau, t_m^0, \ldots, t_m^i$, as in Remark 4.6.*

5 Optimality

In this section, we study the optimal control problems involving differential inclusions and their discrete approximations. This method is important for obtaining the necessary conditions for optimality. We refer the readers to [9, 10, 14, 15, 24, 26] and the bibliography therein for different approaches and various results in this area. In [14, 15, 24, 26], the authors work on a class of differential inclusions under modified one-sided Lipschitz condition and later on Kamke conditions.

However, those conditions are generally violated if the set-valued mapping on the right hand side of the inclusion has unbounded values or is discontinuous. In [9, 10], the authors study the class of sweeping processes governed by polyhedral moving sets. In this paper, motivated by the optimal control problems governed by quasivariational evolution equations which has various applications in ODEs and PDEs problems with nonlocal constraints, quasivariational sweeping processes, parabolic quasivariational inequalities and equations with nondecreasing nonlinearities, we present our study in a more general class of differential inclusions involving maximal monotone operators.

Define $\mathbb{M} := \mathrm{H} \times \mathrm{K}$ and $z = (x, u)$. We consider the optimization problem (P):

$$\text{minimize} \quad J[z] := \varphi(x(T)) + \int_0^T \ell\Big(t, x(t), u(t), \dot{x}(t), \dot{u}(t)\Big) dt \tag{5.54}$$

over the dynamic described by

$$\begin{cases} -\dot{x}(t) \in \partial\Phi(x(t)) + A(t, x(t), u(t)) \text{ for a.e. } t \in [0, T] \\ u(0) = \mathrm{u}_0 \in \mathrm{K} \\ x(0) = \mathrm{x}_0 \in \operatorname{dom} A(0, \cdot, \mathrm{u}_0), \end{cases} \tag{5.55}$$

with the control-state constraint

$$(x(t), u(t)) \in M(t) \quad \forall\, t \in [0, T]. \tag{5.56}$$

We construct the discrete approximations for the optimal solutions to this problem in two separated cases. The first case, we generalize the results obtained in [9, 10, 14, 15] in three directions. First, instead of fixing an optimal solution to problem (P), we only have to choose a minimizing sequence of feasible solutions. Secondly, we consider the control-state constraint (5.56). The most important feature is that we assume that A satisfies $(S1')$, which allow us to apply our results in a wider class of evolution equations, e.g., $A(t, \mathrm{x}, \mathrm{u}) := \nabla\Psi(\mathrm{x}) + N(\mathrm{x}, C(t, \mathrm{u}))$, which cannnot be tackled by previous works. In the second case, we develop a method for constructing a discrete approximation which is *independent* on any information related to the optimal solution to the continuous time problem. It is well known in the theory of differential equations that without coercivity condition, such convergence does not always occur in general. However, by restricting the feasible set of problem (P) on a reasonable domain and by assuming that A satisfies $(S2)$, this convergence holds without imposing any coercivity condition on the cost function (which is always the case for the Mayer problems when $\ell \equiv 0$).

We will assume the following standard assumption on the components of J:

$(O1)$ φ and ℓ are continuous and are bounded from below and, for every $(t, \mathrm{x}, \mathrm{u}) \in [0, T] \times \mathrm{H} \times \mathrm{K}$, $\ell(t, \mathrm{x}, \mathrm{u}, \cdot, \cdot)$ is convex.

Definition 5.1 *Let $\mathbb{H}$ to be a real Hilbert space, let $\mathrm{E} \subset \mathbb{H}$ such that $\mathrm{E} \neq \emptyset$, and consider the optimization problem (WM): $\min_{x \in E} g(x)$. For every $\delta > 0$, $\mathrm{x}_\delta \in E$ is a $\delta-$*weak minimizer of *(WM) if and only if $g(\mathrm{x}_\delta) < g(\mathrm{x}) + \delta\|\mathrm{x} - \mathrm{x}_\delta\|$ for all $\mathrm{x} \in E, \mathrm{x} \neq \mathrm{x}_\delta$.*

Theorem 5.2 *Assume that g is lower-semicontinuous and bounded from below and that $\mathrm{E} \neq \emptyset$ is closed in X. For each $\delta > 0$, there exists a unique $\delta-$weak minimizer of problem (WM).*

Proof. Since E is closed in $\mathbb{H}$, the function $g_{\mathrm{E}} : x \mapsto g(x) + i_{\mathrm{E}}(x)$, where i_{E} is the indicator function of the set E, is lower-semicontinuous. Since g is bounded from below, so is g_{E}. Using Ekeland's weak principle (see [25, Theorem 2.26]) for the function g_{E}, the result follows. △

5.1 Minimizing property of path-following discrete approximations

To construct the discrete problem, we assume that $(O1)$ and all assumptions of Proposition 4.7 hold. By $(O1)$, $\inf J$ exists. Let $\{(\widetilde{x}_n, \widetilde{u}_n)\}_{n\in I\!N}$ be a minimizing sequence of Problem (P) in $\mathcal{C}^1((0,T);\mathbb{M})$, i.e., $(\widetilde{x}_n, \widetilde{u}_n)$ satisfies (5.55) and (5.56) for all $n \in I\!N$ and $J(\widetilde{x}_n, \widetilde{u}_n) \to \inf J$ as $n \to \infty$. By Proposition 4.7, for every $n \in I\!N$, there exists a sequence $(x_m^n, u_m^n)_{m\in I\!N}$ of piecewise linear functions satisfying (4.43) and $(x_m^n, u_m^n) \to (\widetilde{x}_n, \widetilde{u}_n)$ in $W^{1,2}([0,T];\mathbb{M})$ as $m \to \infty$. By taking an appropriate subsequence $(m_n)_{n\in I\!N}$ and defining $(x_n, u_n) = (x_{m_n}^n, u_{m_n}^n)$, we can obtain

$$\|x_n - \widetilde{x}_n\|_{W^{1,2}((0,T);\mathrm{H})} \to 0; \|u_n - \widetilde{u}_n\|_{W^{1,2}((0,T);\mathrm{K})} \to 0 \tag{5.57}$$

as $n \to \infty$.

Now, for every $n \in I\!N$, we consider the discretized problem (P_n):

$$\text{minimize} \quad J_n[\mathrm{z}_n] := \varphi(\mathrm{x}_n^{m_n}) + h_{m_n} \sum_{i=0}^{m_n-1} \ell\Big(t_n^i, \mathrm{x}_n^i, \mathrm{u}_n^i, \frac{\mathrm{x}_n^{i+1} - \mathrm{x}_n^i}{t_n^{i+1} - t_n^i}, \frac{\mathrm{u}_n^{i+1} - \mathrm{x}_n^i}{t_n^{i+1} - t_n^i}\Big)$$
$$+ \sum_{i=0}^{m_n-1} \int_{t_n^i}^{t_n^{i+1}} \Big(\Big\|\frac{\mathrm{x}_n^{i+1} - \mathrm{x}_n^i}{t_n^{i+1} - t_n^i} - \dot{\widetilde{x}}_n(t)\Big\|^2 + \Big\|\frac{\mathrm{u}_n^{i+1} - \mathrm{u}_n^i}{t_n^{i+1} - t_n^i} - \dot{\widetilde{u}}_n(t)\Big\|^2\Big)\, dt,$$

over elements $\mathrm{z}_n := (\mathrm{x}_n^0, \mathrm{u}_n^0, \ldots, \mathrm{x}_n^{m_n}, \mathrm{u}_n^{m_n}) \in \mathbb{M}^{m_n+1}$ satisfying the constraints

$$\begin{cases} -\dfrac{\mathrm{x}_n^{i+1} - \mathrm{x}_n^i}{t_n^{i+1} - t_n^i} \in \partial\Phi(\mathrm{x}_n^{i+1}) + A\big(t_n^i, \mathrm{x}_n^i, \mathrm{u}_n^i\big), & i = 0, \ldots, m_n - 1 \\ \big(\mathrm{x}_n^i, \mathrm{u}_n^i\big) \in M_{\epsilon_n}(t_n^i) & i = 0, \ldots, m_n \\ \mathrm{x}_n^0 = x_0,\ \mathrm{u}_n^0 = u_0. \end{cases} \tag{5.58}$$

and

$$\sum_{i=0}^{m_n-1} \int_{t_n^i}^{t_n^{i+1}} \Big(\Big\|\frac{\mathrm{x}_n^{i+1} - \mathrm{x}_n^i}{t_n^{i+1} - t_n^i} - \dot{\widetilde{x}}_n(t)\Big\|^2 + \Big\|\frac{\mathrm{u}_n^{i+1} - \mathrm{u}_n^i}{t_n^{i+1} - t_n^i} - \dot{\widetilde{u}}_n(t)\Big\|^2\Big) \le 1, \tag{5.59}$$

where $\epsilon_n := \max\limits_{i=0,\ldots,m_n} \{\|x_n(t_n^i) - \widetilde{x}(t_n^i)\|, \|u_n(t_n^i) - \widetilde{u}(t_n^i)\|\}$.

Theorem 5.3 *Suppose that $(O1)$ and the assumptions of Proposition 4.7 hold so that the above constructions of (x_n, u_n) and $(\widetilde{x}_n, \widetilde{u}_n)$ are valid. Recall that $F = \partial\Phi + A$, suppose that $\operatorname{gph} F$ and $\operatorname{gph} M$ are closed and let $(\delta_n)_{n\in I\!N}$ be a positive sequence going to 0. For every $n \in I\!N$, let $\bar{\mathrm{z}}_n \in \mathbb{M}^{m_n+1}$ be a δ_n-weak minimizer of the discrete problem (P_n). Then, up to a subsequence, we have*

$$\Big\|\bar{z}_n(\cdot) - \widetilde{z}_n\Big\|_{W^{1,2}([0,T];\mathbb{M})} \to 0 \ \text{ and } \ J_n(\bar{z}_n) \to \inf J \text{ as } n \to \infty, \tag{5.60}$$

where $\bar{z}_n(\cdot)$ is piecewise linearly extended function associated to $\bar{\mathrm{z}}_n$.

Proof. We recall the sequence $\{z_n\}_{n\in I\!N} = \{(x_n, u_n)\}_{n\in I\!N}$ defined in the beginning of Section 5.1. Note that, for every $n \in I\!N$, z_n is feasible to (P_n) by construction and, since $\operatorname{gph}\partial\Phi$, $\operatorname{gph} F$, and $\operatorname{gph} M$ are closed, the constraint set is closed. Hence, for every $n \in I\!N$, Theorem 5.2 guarantees the existence of the unique δ_n-minimizer to (P_n), denoted by $\bar{\mathrm{z}}_n$. Moreover, denote by $\bar{z}_n(\cdot)$ the piecewise linearly extended function associated to $\bar{\mathrm{z}}_n$. The constraints (5.58) and (5.59) guarantee that $(\bar{z}_n - \widetilde{z}_n)_{n\in I\!N}$ is uniformly bounded in $W^{1,2}([0,T];\mathbb{M})$. Now, suppose by contradiction that the first limit in (5.60) does not hold, i.e., the limit along a subsequence therein (no relabelling) is some $\gamma > 0$. By the weak compactness of the unit ball in $L^2 := L^2([0,T];\mathbb{M})$ there exists $(v(\cdot), w(\cdot)) \in L^2([0,T];\mathbb{M})$ and yet another subsequence of $\{\bar{z}_n(\cdot)\}$ such that

$$\big(\dot{\bar{x}}_n(\cdot), \dot{\bar{u}}_n(\cdot)\big) \to \big(v(\cdot), w(\cdot)\big) \text{ weakly in } L^2.$$

Define now the absolutely continuous function $z(\cdot) := (x(\cdot), u(\cdot))\colon [0,T] \to \mathrm{H} \times \mathrm{K}$ by

$$z(t) = (x(t), u(t)) := (x_0, u_0) + \int_0^t \big(v(s), w(s)\big)\, ds, \quad t \in [0,T],$$

for which $\dot{z}(t) = (v(t), w(t))$ almost everywhere on $[0,T]$. The Mazur Weak Closure Theorem allows us to find a sequence of convex combinations of $(\dot{\bar{x}}_n(\cdot), \dot{\bar{u}}_n(\cdot))$, which converges to $(v(\cdot), w(\cdot))$ strongly in L^2 and thus almost everywhere on $[0,T]$ along a subsequence. The closedness of $\operatorname{gph}\partial\Phi$, $\operatorname{gph} F$ and $\operatorname{gph} M$ guarantee that $z(\cdot)$ satisfies (5.55)-(5.56). Moreover, the convexity of ℓ in the velocities and its lower-semicontinuity yield

$$\int_0^T \ell\big(t, x(t), u(t), \dot{x}(t), \dot{u}(t)\big)\, dt \le \liminf_{n\to\infty} h_{m_n} \sum_{i=0}^{m_n-1} \ell\Big(t_n^i, \bar{\mathrm{x}}_n^i, \bar{\mathrm{u}}_n^i, \frac{\bar{\mathrm{x}}_n^{i+1} - \bar{\mathrm{x}}_n^i}{t_n^{i+1} - t_n^i}, \frac{\bar{\mathrm{u}}_n^{i+1} - \bar{\mathrm{u}}_n^i}{t_n^{i+1} - t_n^i}\Big). \quad (5.61)$$

Thus the passage to the limit in the cost functional of (P_m) and the definition of $\gamma > 0$ show that

$$J[z] + \gamma = \varphi\big(x(T)\big) + \int_0^T \ell\big(t, x(t), u(t), b(t), \dot{x}(t), \dot{u}(t), \dot{b}(t)\big)\, dt + \gamma \le \liminf_{n\to\infty} J_n[\bar{\mathrm{z}}_n]. \quad (5.62)$$

It follows from (5.57) and $(O1)$ that $J_n[z_n] \to \inf J$ as $n \to \infty$. Since, for every $n \in I\!N$, z_n is feasible to (P_n) and $\bar{\mathrm{z}}_n$ is its δ_n-minimizer, we obtain

$$J_n[\bar{\mathrm{z}}_n] \le J_n[z_n] + \delta_n \|z_n - \bar{\mathrm{z}}_n\|_{\mathbb{M}^{m_n+1}}.$$

Passing to the limit and using the fact that $J_n[z_n] \to \inf J$, we obtain

$$\limsup_{n\to\infty} J_n[\bar{\mathrm{z}}_n] \le \inf J. \quad (5.63)$$

Finally, by combining (5.62) and (5.63), we have

$$J[z] + \gamma \le \inf J,$$

which is a contradiction. This justifies the validity of (5.60) and $(O1)$ yields the last assertion. $\triangle$

Remark 5.4 If the function ℓ does not depend on $\dot{u}$, instead of assuming the convergence in $\mathcal{C}^1((0,T);\mathrm{K})$ of u_n to $\widetilde{u}_n$ and the assumption $(S1')$ on A (as in Theorem 5.3), we only have to assume the convergence in $\mathcal{C}((0,T);\mathrm{K})$ of u_n to $\widetilde{u}_n$ and the assumption $(S1)$ on A. Indeed, these assumptions allow us to apply Theorem 4.4 and construct, in the same way as in Section 5.1, a sequence of piecewise linear functions (x_n, u_n) satisfying (5.55) and

$$\|x_n - \widetilde{x}_n\|_{W^{1,2}((0,T);\mathrm{H})} \to 0; \|u_n - \widetilde{u}_n\|_{\mathcal{C}((0,T);\mathrm{K})} \to 0 \tag{5.64}$$

as $n \to \infty$. Therefore, by following the proof of Theorem 5.3 we obtain that any sequence of piecewise linear functions $(\bar{z}_n)_{n\in I\!N}$ constructed from δ_n-weak minimizers $\{\bar{\mathrm{z}}_n\}_{n\in I\!N}$ of (P_n) with $\delta_n \to 0$ satisfy, up to a subsequence,

$$\|\bar{x}_n - \widetilde{x}_n\|_{W^{1,2}([0,T];\mathrm{H})} \to 0 \quad ; \|\bar{u}_n - \widetilde{u}_n\|_{\mathcal{C}((0,T);\mathrm{K})} \to 0, \;\; J_n(\bar{z}_n) \to \inf J \quad \text{as } n \to \infty. \tag{5.65}$$

5.2 Direct method

We assume that $\dim \mathrm{K} < \infty$ and consider the Mayer problem (Q):

$$\text{minimize} \quad J[x,u] := \varphi(x(T))$$

governed by the dynamic

$$\begin{cases} -\dot{x}(t) \in F\big(t, x(t), u(t)\big) \text{ a.e. } t \in [0,T]; \\ x(0) = x_0; u(0) = u_0, \end{cases}$$

as well as the geometric constraint

$$u \in \mathbb{U} \subset \{u \in \mathcal{C}([0,T];\mathrm{K}) | \|u\|_\infty \le \mathcal{L}\}.$$

for a fixed number $\mathcal{L} > 0$. The discretized problem (Q_m) is stated as follows:

$$\text{minimize} \quad J_m[\mathrm{z}_m] := \varphi(\mathrm{x}_m^m)$$

over elements $\mathrm{z}_m := (\mathrm{x}_m^0, \mathrm{u}_m^0, \ldots, \mathrm{x}_m^m, \mathrm{u}_m^m) \in \mathbb{M}^{m+1}$ such that

$$\begin{cases} -\dfrac{\mathrm{x}_m^{i+1} - \mathrm{x}_m^i}{t_m^{i+1} - t_m^i} \in F(t_m^{i+1}, \mathrm{x}_m^{i+1}, \mathrm{u}_m^{i+1}) \quad (\forall i = 0, \ldots, m-1) \\ \mathrm{x}_m^0 = \mathrm{x}_0; \mathrm{u}_m^0 = \mathrm{u}_0 \\ u_m \in \mathbb{U}, \end{cases} \tag{5.66}$$

where u_m is the piecewise linear extension of $(\mathrm{u}_m^0, \ldots, \mathrm{u}_m^m)$. Note that, by denoting $\lambda_m = t_m^{i+1} - t_m^i$ for every $i = 1, \ldots, m$, (5.66) can be written equivalently as

$$\begin{cases} (\forall i = 0, \ldots, m-1) \quad \mathrm{x}_m^{i+1} := J_{\lambda_m}^{t_m^{i+1}, \mathrm{u}_m^{i+1}}(\mathrm{x}_m^i) \\ \mathrm{x}_m^0 = x_0; \mathrm{u}_m^0 = u_0. \end{cases} \tag{5.67}$$

Theorem 5.5 *Assume that $(O1)$ holds, F satisfies conditions $(E1)$, $(E2)$, $(E3)$ and* $\operatorname{gph} F$ *is closed. Assume further that*

$$\sup_{u\in\mathbb{U}} \max\{\gamma(u), \theta(u), \alpha(u,r), \beta(u,r)\} < \infty \tag{5.68}$$

for all $r > 0$. Then for any sequence $\{\bar{z}_n\}_{n\in I\!N}$ of δ_n-weak minimizer of the discrete problems (Q_n) where $\delta_n \to 0$ as $n \to \infty$, there exists a subsequence that converges in $W^{1,2}\big([0,T];\mathrm{H}\big) \times \mathcal{C}\big([0,T];\mathrm{K}\big)$ to some solution $(\bar{x}, \bar{u})$ to problem (Q).

Proof. First, since the graph of F and $\mathbb{U}$ is closed, the feasible set of problem (Q_m) is closed. Since φ is continuous and bounded from below, a δ_m-minimizer $\bar{z}_m := (\bar{x}_m, \bar{u}_m)$ of (Q_m) exists. Next, $\bar{u}_m$ is uniformly bounded in $\mathcal{C}([0,T];\mathrm{K})$ because $\bar{u}_m \in \mathbb{U} \subset \{u \in \mathcal{C}([0,T];\mathrm{K}) | \|u\|_\infty \leq \mathcal{L}\}$. Using condition (5.68) and Theorem 3.8, $\{\bar{x}_m\}_{m\in\mathbb{N}}$ is uniformly bounded in $W^{1,2}\big([0,T];\mathrm{H}\big)$. But $\mathbb{U}$ is bounded in K, and so $\{\bar{u}_m\}_{m\in\mathbb{N}}$ is uniformly bounded in $L^2\big([0,T];\mathrm{K}\big)$. By the weak compactness of the unit ball in $L^2 := L^2([0,T];\mathrm{H})$, there exists $v(\cdot) \in L^2([0,T];\mathrm{H})$ and yet another subsequence of $\{\bar{x}_m(\cdot)\}$ such that $\dot{\bar{x}}_m(\cdot) \to v(\cdot)$ weakly in $L^2([0,T];\mathrm{H})$. Having $\{\bar{u}_m\}_{m\in\mathbb{N}} \subset \mathbb{U}$, up to subsequence, $\bar{u}_m(t)$ converges weakly in K to $\bar{u}(t)$ for almost every $t \in [0,T]$. Define now the absolutely continuous function $\bar{x}\colon [0,T] \to \mathrm{H} \times \mathrm{K}$ by $\bar{x}(t) := \mathrm{x}_0 + \int_0^t v(s)\,ds$ for all $t \in [0,T]$. The Mazur Weak Closure Theorem allows us to find a sequence of convex combinations of $\dot{\bar{x}}_m(\cdot)$, which converges to $v(\cdot)$ strongly in L^2 and thus almost everywhere on $[0,T]$ (along a subsequence). The closedness of $\operatorname{gph} F$ guarantees that $\bar{z}(\cdot)$ satisfies (4.37). Moreover, the strong convergence of $\overline{\mathrm{x}}_m^m$ to $\bar{x}(T)$ guarantees that

$$J[\bar{x},\bar{u}] \leq \liminf_{m\to\infty} J_m[\overline{\mathrm{z}}_m].$$

Then, for any feasible solution (x,u) of problem (Q), following the proof of Theorem 3.8, there exists a sequence $\{(\mathrm{x}_m,\mathrm{u}_m)\}_{m\in\mathbb{N}}$ feasible to (Q_m) that converges uniformly to x up to a subsequence. Since $\overline{\mathrm{z}}_m$ is δ_m-weak minimizer to problem (Q_m),

$$\liminf_{m\to\infty} J_m[\overline{\mathrm{z}}_m] \leq \liminf_{m\to\infty} J_m[\mathrm{z}_m] + \delta_m \|\mathrm{z}_m - \bar{z}_m\|_{\mathbb{M}^{k+1}} = \liminf_{m\to\infty} J_m[\mathrm{z}_m]. \tag{5.69}$$

But φ is continuous, and so

$$\lim_{m\to\infty} J_m[\mathrm{z}_m] = J[x,u].$$

Combining the three inequalities above, we get $J[\bar{x},\bar{u}] \leq J[x,u]$ and conclude. △

6 Application to Electrical Circuits with Ideal Diodes

In this section we apply our results to the case of electrical circuits with ideal diodes. We discuss the circuit considered in [1, Section 3.4]. By applying Kirchhoff's laws for ideal diode model, the following inclusion is obtained

$$-\dot{x}(t) \in \frac{R}{L}x(t) + N(x(t);C(t)), \tag{6.70}$$

where $C(t) := [u(t), +\infty)$. This fits our model (1.2) where $F(t,\mathrm{x},\mathrm{u}) := \partial\Phi(\mathrm{x}) + N(\mathrm{x};\mathrm{C}+\mathrm{u})$, $\Phi(\mathrm{x}) := \frac{R}{2L}|\mathrm{x}|^2$ and $C := [0,+\infty)$.

We now show that our results in Theorem 3.8, Theorem 4.8 and Theorem 5.5 can be applied for this particular case. To do that, we assume, further, that u is a continuous function on $[0,T]$ and is of bounded variation.

First of all, by using Proposition 3.1, $(E1)$ is satisfied with $\kappa_C(r) = 1$, $L_u = \mathcal{L}$ and $\alpha(u,r) := (1+2L_\Phi^r)(\mathcal{L}+1)$. Next, $(E2)$ is satisfied with $\beta(u,r) := L_\Phi^r$ by Proposition 3.3. Proposition 3.5 gives $\theta(u) := |\mathrm{u}_0| + T\mathcal{L}$. Moreover, $\xi(u,0) := 2(\theta(u)+1) + \beta(u,\theta(u)+1)$ by Theorem 3.8. Finally, Proposition 3.5 implies $(E3)$ is satisfied with $\gamma(u) := \mathrm{var}(u;[0,T]) + 2\sqrt{L_\Phi^{\xi(u,0)}}\sqrt{T\mathrm{var}(u;[0,T])} = \mathcal{L}T + 2T\sqrt{L_\Phi^{\xi(u,0)}\mathcal{L}}$. By Theorem 3.8, since $u \in \mathcal{C}([0,T];\mathbb{R})$ is of bounded variation, for a given $\mathrm{x}_0 \in [u(0),+\infty)$, there exists a solution $\bar{x} \in W^{1,2}\big([0,T];\mathbb{R}\big)$ of (6.70) such that $\bar{x}(0) = \mathrm{x}_0$.

Now, we check that all assumptions of Theorem 4.8 are satisfied. The assumption (4.52) holds with $\bar{f} = -\dot{\bar{x}} - \frac{R}{L}\bar{x}$. By Proposition 4.3, $(S2)$ is satisfied. Finally, Proposition 3.5 and $\mathbb{U}_2 := \{u \in \mathrm{Lip}([0,T];\mathrm{K}) | \mathrm{Lip}(u) \leq \mathcal{L}\} \subset \mathbb{U}$ imply that condition (5.68) holds. As a result, we can apply Theorem 5.5 for (6.70).

Optimality results can also be obtained using the results in Section 5, if the given objective functional is reasonably regular.

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