MONOTONICITY BEYOND MINTY AND KATO ON LOCALLY CONVEX SPACES

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Abstract. This paper is a contribution to the theory of monotone operators in topological vector spaces. On the one hand, we provide new results concerning topological and geometric properties of monotone operators satisfying mild continuity assumptions. In particular, we give fairly general conditions on the operator to become single-valued, to be closed and maximal. A fundamental tool is a generalization of the well-known Minty's Lemma that is interesting in its own right and, surprisingly, remains true for general topological vector spaces. As a consequence of Minty's, we obtain an extension of a rather remarkable theorem of Kato for multi-valued mappings defined on general locally convex spaces.

Introduction

Monotone operators have been extensively studied for the last fifty years in terms of their structural properties ([7, 8, 15, 17, 18, 19, 25, 27, 29]), their connection with certain dynamical systems ([4, 5, 10, 24, 28]) and the solution of functional equations ([1, 9, 20, 22, 23]) arising, for instance, in convex optimization and equilibrium problems, partial differential equations and variational inequalities. A recent survey on the history of monotone operators has been carried out by Borwein [3].

In the last fifty years, the study of monotone operators has been mostly developed for reflexive Banach spaces, and especially for Hilbert spaces. However, many equilibrium and optimization problems escape this setting. For instance, even the most classical optimal control problems, where the cost functional is convex and the dynamics is given by a system of linear ordinary differential equations, the natural functional setting is the nonreflexive space of continuous (or continuously differentiable) functions defined on a given interval. As far as we know, this class of problems has not been addressed using the monotone operator theory approach. In fact, neither from a theoretical point of view, nor from the numerical one. The present research attempts to set the basis of the theoretical background − in terms of some topological, geometrical and algebraic properties − and laying the foundations for future applications. More precisely, we study various properties of monotone operators defined on topological vector spaces, as well as locally convex spaces, under suitable continuity conditions.

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The paper is organized as follows: In Section 1 we present and extension of Minty's Lemma (see [19, 20, 2, 9, 11]) to topological vector spaces, and more general operators and domains. This result, namely Lemma 1, is the key to establishing several new properties of monotone operators in locally convex spaces. Section 2 contains conditions under which a monotone operator is single-valued and the points of its domain where this property holds. Next, in Section 3, we study maximality and $\mathcal{D}$-maximality. The main results of this section extend a remarkable result of Kato to locally convex spaces (see Theorem 11 and also Corollary 13). Section 4 contains several results allowing to better understand the relationship between maximality and different types of continuity. Finally, in Section 5, we provide some closedness results concerning both the values and the graph of the operator.

We should mention that throughout this work, X will represent a real vector space.

1. An extension of Minty's Lemma

Let X be a topological vector space with topological dual X^* and denote by $\langle x^*, x\rangle$ the action of $x^* \in X^*$ on $x \in X$. For the purpose of this paper, an *operator* A defined on X with set-values in X^* shall be denoted by $A : X \to 2^{X^*}$. Its *effective domain* is the set

$$\mathcal{D}(A) = \{x \in X : Ax \neq \emptyset\}.$$

An operator $A : X \to 2^{X^*}$ is *monotone* if

$$\langle x^* - y^*, x - y\rangle \geq 0$$

for all $x,\ y \in \mathcal{D}(A)$ and all $x^* \in Ax$, $y^* \in Ay$.

An operator $A : X \to 2^{X^*}$ is *lower-demicontinuous* if for every $z_0 \in \mathcal{D}(A)$ and every weak* open set $\mathcal{O}$ in X^* with $Az_0 \cap \mathcal{O} \neq \emptyset$, there exists a neighborhood U of z_0 such that

$$Az \cap \mathcal{O} \neq \emptyset \quad \text{for every} \quad z \in \mathcal{D}(A) \cap U.$$

The operator $A : X \to 2^{X^*}$ is *upper-demicontinuous* if for every $z_0 \in \mathcal{D}(A)$ there exists a weak* open set $W \subset X^*$ with $Az_0 \subset W$, such that for every neighborhood U of z_0 one has

$$Az \subset W \quad \text{for every} \quad z \in U.$$

In addition, A is *lower-demicontinuous (resp. upper-demicontinuous) on finite dimensional subspaces* if the restriction of A to $\mathcal{D}(A) \cap Y$ is lower-demicontinuous (resp. upper-demicontinuous) for any finite dimensional subspace Y of X. Finally, A is *lower-hemicontinuous* (resp. *upper-hemicontinuous*) if it is lower-demicontinuous (resp. upper-demicontinuous) on line segments. For the single-valued case, the definitions mentioned above coincide with the concepts of demicontinuity and hemicontinuity introduced by Minty [20] and Browder [6].

As we shall see, lower-hemicontinuous monotone operators have interesting topological and geometrical properties, even in topological vector spaces. We begin by establishing an extension of a key result due to Minty (see Lemma 1), that is interesting in its own right.

Let $D \subset X$ be a nonempty set. A point $z \in X$ is *densely sequentially approachable within* D if there exists a subset $D_z \subset X$, whose closure contains D, such that for all $x \in D_z$, there exists a sequence of positive numbers $(t_n) \subset \mathbb{R}_+$, decreasing to 0 with

$$z + t_n(x - z) \in D \quad \text{for all } n \in \mathbb{N}.$$

This notion is weaker than one of *being surrounded densely by* D, introduced by Minty in [20]. We denote by $SA(D) \subset D$ the set of points of D that are densely sequentially approachable within D. The set D is *densely sequentially approachable* (SA, for short) if $D = SA(D)$.

Next, we give some examples of this new notion.

(1) Clearly, $\mathrm{int}(D) \subset SA(D)$. As a consequence, every open set is SA. The same is true for (algebraically) relatively open sets.
(2) Every *quasi dense* set is SA. Recall that, following [15], a set D is quasi-dense if for each $z \in D$ there exists a dense subset M_z of X such that for each $v \in M_z$, $z + tv \in D$ for all sufficiently small $t > 0$.
(3) If D is star-shaped with center z, then $z \in SA(D)$. Therefore, every convex set is SA.
(4) For the *spiral* $D_0 = \{(e^{-t}\cos(t), e^{-t}\sin(t)) \in \mathbb{R}^2 : t > 0\}$, the origin is sequentially approachable within D_0.
(5) Let D_0^c be the complement of the set D_0 defined above. It is a *thick spiral* containing the origin. Clearly $(0,0) \in SA(D_0^c)$. The remaining points of D_0^c are interior, thus also belong to $SA(D_0^c)$. Therefore, this rather exotic set is SA.

Lemma 1. *Let X be a topological vector space, let $A : X \to 2^{X^*}$ be a monotone and lower-hemicontinuous operator, and let $z \in SA(\mathcal{D}(A))$. The following are equivalent:*

(i) $\langle x^*, x - z\rangle \geq 0$ *for all* $x \in \mathcal{D}(A)$ *and all* $x^* \in Ax$*;*
(ii) $\langle z^*, x - z\rangle \geq 0$ *for all* $x \in \mathcal{D}(A)$ *and all* $z^* \in Az$.

Proof. Due to the monotonicity of A, it is enough to show that (i) implies (ii). Let $z^* \in Az$. Since z is densely sequentially approachable within $\mathcal{D}(A)$, there exists a set $D_z \subset X$ with $\mathcal{D}(A) \subset \overline{D_z}$ and, for all x in D_z, there exists a sequence of positive numbers $(t_n) \subset \mathbb{R}_+$ such that $t_n \to 0$ as $n \to \infty$ and $y_n = z + t_n(x - z) \in \mathcal{D}(A)$ for $n \geq 1$.

Take $x \in D_z$ and for $\eta > 0$ define

$$\mathcal{O} = \{\zeta^* \in X^* : \langle \zeta^* - z^*, x - z\rangle < \eta\},$$

which is open in X^* for the weak* topology. By the continuity assumption on A applied to $seg[z, x] := \{z + t(x - z) \ : \ t \in [0, 1]\}$, we deduce the existence of $y_n^* \in Ay_n \cap \mathcal{O}$ for all n sufficiently large. By assumption we have $\langle y_n^*, y_n - z\rangle \geq 0$. On the other hand, $x - z = \frac{1}{t_n}(y_n - z)$. Whence

$$\begin{aligned}
\langle z^*, x - z\rangle &= \langle z^* - y_n^*, x - z\rangle + \langle y_n^*, x - z\rangle \\
&= \langle z^* - y_n^*, x - z\rangle + \frac{1}{t_n}\langle y_n^*, y_n - z\rangle \\
&> -\eta.
\end{aligned}$$

Since η is arbitrary we must have

$$\langle z^*, x - z\rangle \geq 0 \qquad \text{for all} \quad x \in D_z. \tag{1}$$

Since $\mathcal{D}(A) \subset \overline{D_z}$, from (1) we can conclude the desired inequality $\langle z^*, x - z\rangle \geq 0$ for all $x \in \mathcal{D}(A)$. □

Remark 2. Lemma 1 extends Minty's (see [19, 20, 2, 9, 11]) in various directions: First, from reflexive Banach spaces to topological vector spaces; second, our extension holds for set-valued lower-hemicontinuous mappings rather than single-valued mappings that are continuous on finite dimensional subspaces; and finally, we consider more general type of domains, beyond the convex ones.

2. Single-valuedness

An important consequence of the preceding result is presented below and it concerns the fact that under certain assumptions, the operator is always single-valued.

Given a nonempty set $D \subset X$, the *normal cone* of D at $x \in D$, denoted by $N_D(x)$, is the set

$$N_D(x) = \{x^* \in X^* \ : \ \langle x^*, x - u\rangle \geq 0 \ \text{ for all } \ u \in D\}.$$

For $x \notin D$ the normal cone of D at x is the empty set. Observe that $0 \in N_D(x)$ for every $x \in D$. Moreover, if $x \in \mathrm{int}(D)$, then $N_D(x) = \{0\}$. Finally, note that if D is a quasi-dense set, then $N_D(x) = \{0\}$ for every $x \in D$.

The following result establishes single-valuedness whenever the normal cone is *pointed*[1]:

Theorem 3. *Let X be a topological vector space and let $A : X \to 2^{X^*}$ be a lower-hemicontinuous monotone operator. Then, A is single-valued at every point $z \in SA(\mathcal{D}(A))$ such that $N_{\mathcal{D}(A)}(z)$ is pointed.*

Proof. Let $z \in SA(\mathcal{D}(A))$ and let $z_1^*, z_2^* \in Az$. The monotonicity of A implies

$$\langle x^* - z_1^*, x - z\rangle \geq 0$$

for each $x \in \mathcal{D}(A)$ and $x^* \in Ax$. By applying Lemma 1 to the operator $B : X \to 2^{X^*}$ defined by $Bx = Ax - \{z_1^*\}$, we deduce that

$$\langle z_2^* - z_1^*, x - z\rangle \geq 0$$

for each $x \in \mathcal{D}(A)$ and therefore, $z_1^* - z_2^* \in N_{\mathcal{D}(A)}(z)$. With the same arguments, we can prove that $z_2^* - z_1^* \in N_{\mathcal{D}(A)}(z)$. If $N_{\mathcal{D}(A)}(z) \cap \big[- N_{\mathcal{D}(A)}(z)\big] = \{0\}$, then $z_1^* = z_2^*$. □

An immediate consequence is the following new result concerning single-valuedness of operators defined on topological vector spaces in the interior of their domains:

Corollary 4. *Let X be a topological vector space and let $A : X \to 2^{X^*}$ be a monotone and lower-hemicontinuous operator. Then A is a single-valued mapping on int$(\mathcal{D}(A))$.*

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[1]A cone K is pointed if $K \cap [-K] = \{0\}$.

Remark 5. Corollary 4 is an extension of [13, Corollary 2.2] from Hilbert to topological vector spaces. In addition, our result is formulated under a weaker continuity assumption over more general type of domain and using only monotonicity property (not strong monotonicity). Also, it represents a X-to-X^* counterpart of [13, Theorem 2.1].

As in Corollary 4, we obtain an even more general result for quasi-dense domains. Notice that these sets are SA and the normal cone is reduced to $\{0\}$ at every point.

Corollary 6. *Let X be a topological vector space and let $A : X \to 2^{X^*}$ be a monotone and lower-hemicontinuous operator with quasi-dense domain $\mathcal{D}(A)$. Then, A is a single-valued mapping.*

Remark 7. In [30], [17], and [16], it is studied the set of points where a monotone operator is not single-valued, proving that this set has empty interior (when X is a separable Banach space) and has Lebesgue measure zero (when X is fine-dimensional). The above corollaries extend the mentioned results for more general spaces, under lower-hemicontinuity assumption for the operator.

3. Sufficient conditions for maximality

A monotone operator $A : X \to 2^{X^*}$ is said to be *maximal* if its graph is not properly contained in the graph of any other monotone operator. In other words, whenever the condition

$$\langle x^* - z^*, x - z\rangle \geq 0 \quad \text{ for all } x \in \mathcal{D}(A) \text{ and } x^* \in Ax,$$

implies $z \in \mathcal{D}(A)$ and $z^* \in Az$. A typical example of a maximal monotone operator is the subdifferential of a continuous convex function everywhere defined on a topological vector space [21, Theorem 2].

More generally, a monotone operator A is $\mathcal{D}$-*maximal* (see Browder [7]) if the condition

$$(z, z^*) \in \mathcal{D}(A) \times X^* \quad \text{and} \quad \langle x^* - z^*, x - z\rangle \geq 0 \quad \text{ for all } x^* \in Ax \text{ with } x \in \mathcal{D}(A),$$

implies $z^* \in Az$.

Proposition 8. *Let X be a topological vector space and let $A : X \to 2^{X^*}$ be a monotone and lower-hemicontinuous operator with quasi dense-domain $\mathcal{D}(A)$. Then A is a single-valued $\mathcal{D}$-maximal operator.*

Proof. Observe that by Corollary 6, the operator A must be single-valued. Now, let $z \in \mathcal{D}(A)$ and $z^* \in X^*$ such that

$$\langle Ax - z^*, x - z\rangle \geq 0$$

for all $x \in \mathcal{D}(A)$. We shall prove that $z^* = Az$. Indeed, as in the proof of Theorem 3, Lemma 1 implies

$$\langle Az - z^*, x - z\rangle \geq 0$$

for all $x \in \mathcal{D}(A)$. Once again, the quasi density of $\mathcal{D}(A)$ allow us to conlcude $Az = z^*$. □

In particular, we have the following result.

Corollary 9. *Let X be a topological vector space and let $A : X \to 2^{X^*}$ be monotone and lower-hemicontinuous with $\mathcal{D}(A) = X$. Then A is maximal monotone.*

Remark 10. Corollary 9 extends [12, Corollary 2.7 of Chapter V], and [3, Proposition 1]. The two cited results assume that the operator is single-valued, which we happen to prove of being unnecessary. Moreover, the first result is proved for Banach spaces.

4. Local boundedness, $\mathcal{D}$-maximality and continuity

Recall that an operator A is *locally bounded* at x_0 if there exists a neighborhood U of x_0 such that the set

$$A(U) = \cup\{A(x) : x \in U\}$$

is a relatively weak*-compact set of X^*.

As a consequence of some previous results, we obtain the following:

Theorem 11. *Let X be a locally convex Hausdorff space and let $A : X \to 2^{X^*}$ be a monotone and lower-hemicontinuous operator with a quasi-dense domain $\mathcal{D}(A)$. If A is locally bounded, then A is a single-valued demicontinuous $\mathcal{D}$-maximal monotone operator.*

Proof. First of all, due to Corollary 6 and Proposition 8, we derive that A is a single-valued $\mathcal{D}$-maximal monotone operator. Then by Theorem 1 of [18], we conclude that A is demicontinuous on $\mathcal{D}(A)$. □

As a consequence of Theorem 11, we derive a result for Fréchet spaces, but first, we need the following proposition.

Proposition 12. *Let X be a Fréchet space and let $A : X \to X^*$ be a hemicontinuous monotone operator with quasi-dense domain $\mathcal{D}(A)$. Then A is locally bounded.*

Proof. The proof is essentialy included in the proof of the theorem of [14]. □

Corollary 13. *Let X be a Fréchet space and let $A : X \to 2^{X^*}$ be monotone and lower-hemicontinuous operator with a quasi-dense domain en A is $\mathcal{D}$-maximal and demicontinuous on $\mathcal{D}(A)$.*

Proof. From Corollary 6, A is single-valued on $\mathcal{D}(A)$. Since X is a Fréchet space, A is locally bounded (from Proposition 12). Consequently, by Theorem 11, A is demicontinuous and $\mathcal{D}$-maximal on $\mathcal{D}(A)$. □

Remark 14. Theorem 11, and especially Corollary 13, extend Theorem 1 of Kato [15] from Banach spaces to locally convex spaces and Fréchet spaces respectively.

In this section, we present several results that relates $\mathcal{D}$-maximal monotonicity with continuity properties on the operator. Particularly, at the end of the section, we address the question under what assumptions, monotone operators are demicontinuous single-valued.

Theorem 15. *Let X be a locally convex Hausdorff space and $\mathcal{D} \subset X$ a nonempty set. Let $A : X \to 2^{X^*}$ be a $\mathcal{D}$-maximal monotone operator, which is locally bounded on finite dimensional subspaces. Then A is upper-hemicontinuous on $\mathcal{D}$.*

Proof. Let $x_0 \in \mathcal{D}(A)$. Suppose A is not upper-hemicontinuous at x_0. This means, for some $z \in \mathcal{D}(A)$, there exists a weak* open subset $W \subset X^*$, with $Ax_0 \subset W$, such that for each neighborhood U of x_0 relative to the segment $seg[z, x_0]$, there exist $x \in U$ and $x^* \in Ax$ with $x^* \notin W$.

First, since A is locally bounded on finite dimensional subspaces, there exists a neighborhood V of x_0 relative to $seg[z, x_0]$ such that $A(V)$ is a relatively weak*-compact set of X^*.

Second, since the segment $seg[z, x_0]$ is contained in a finite dimensional subspace, and there is only one topology compatible with the structure of locally convex topological Hausdorff space, we may take a sequence of neighborhoods $(V_n) \subset V$ of x_0 relative to $seg[z, x_0]$, such that for any sequence $(z_n) \subset V$ with $z_n \in V_n$, one has $z_n \to x_0$. Therefore, in view of the first paragraph, we can select a sequence $(x_n) \subset V$ with $x_n \in V_n$, for which there exists $x_n^* \in Ax_n \subset A(V)$ such that $x_n^* \notin W$ for each $n \in \mathbb{N}$. Clearly, from the choice of neighborhoods V_n, $x_n \to x_0$.

From the weak* relative compactness of $A(V)$, we can obtain a subnet (x_i^*) of (x_n^*) convergent, in the weak* topology, to some $x_0^* \notin W$ and a subnet (x_i) of (x_n), with $x_i^* \in Ax_i$. Since $x_n \to x_0$, we also have $x_i \to x_0$. Then, the monotonicity of A implies that $\langle x_i^* - x^*, x_i - x\rangle \geq 0$ for all $x^* \in Ax$ and all $x \in \mathcal{D}(A)$.

We conclude

$$\langle x_0^* - x^*, x_0 - x\rangle \geq 0 \quad \text{for all} \quad x^* \in Ax \;\; \text{and for all} \quad x \in \mathcal{D}(A).$$

Therefore, the $\mathcal{D}$-maximal monotonicity of A implies that $x_0^* \in Ax_0$, which contradicts the fact that $x_0^* \notin W$. Consequently, A is upper-hemicontinuous on $\mathcal{D}(A)$. □

For the single-valued case, we obtain this interesting fact, that appears to be new, and is a direct consequence of Theorem 15.

Corollary 16. *Let $\mathcal{D}$ be a subset of a locally convex space X and let $A : \mathcal{D} \to X^*$ be a $\mathcal{D}$-maximal monotone operator, which is locally bounded on finite dimensional subspaces. Then A is hemicontinuous on $\mathcal{D}$.*

The following is a consequence of a Rockafellar [27] and Kravvaritis [18] results.

Theorem 17. *Let X be a locally convex (real) Hausdorff space and let $A : X \to 2^{X^*}$ be a $\mathcal{D}$-maximal monotone operator. Suppose that A is locally bounded at some $x \in G := int(\overline{\mathcal{D}(A)})$. Then A is upper-demicontinous and locally bounded on $G \cap \mathcal{D}(A)$.*

Proof. By [27, Corollary 2.2], we obtain that A is locally bounded on all of G. Since A is $\mathcal{D}$-maximal, it is also $(G \cap \mathcal{D}(A))$-maximal. Hence, by [18, Theorem 1], we conclude that A is upper demicontinuous on $G \cap \mathcal{D}(A)$. □

Remark 18. We notice that Theorem 17 extends Theorem 2 of [17] for Banach spaces to locally convex Hausdorff spaces.

A direct consequence of Theorem 17 is the following corollary.

Corollary 19. *Let X be a locally convex Hausdorff space and let $A : X \to 2^{X^*}$ be a lower-hemicontinuous and $\mathcal{D}$-maximal monotone operator. Suppose that A is locally bounded at some $x \in int(\overline{\mathcal{D}(A)})$. Then A is single-valued and demicontinuous on $int(\mathcal{D}(A))$.*

Proof. Due to Corollary 4, A is single-valued on $\mathrm{int}(\mathcal{D}(A))$, and consequently, by Theorem 17, A is demicontinuous on $\mathrm{int}(\mathcal{D}(A))$. □

We complete this section by replacing the quasi-density of $\mathcal{D}(A)$ by being open and obtain an extension to arbitrary locally convex spaces of the Theorem in [14].

Corollary 20. *Let X be a locally convex Hausdorff space and let $A : X \to 2^{X^*}$ be monotone and lower-hemicontinuous operator with open domain $\mathcal{D}(A)$. Suppose A is locally bounded at some $x \in \mathcal{D}(A)$. Then A is a demicontinuous $\mathcal{D}$-maximal monotone operator, which is locally bounded on $\mathcal{D}(A)$.*

Proof. First of all, due to Proposition 8, we derive that A is a $\mathcal{D}$-maximal monotone operator. Then as a consequence of Theorem 17 combined with Corollary 19, we obtain that A is demicontinuous on $\mathcal{D}(A)$. The locally boundedness on $\mathcal{D}(A)$ follows from [27, Corollary 2.2]. □

Kato [15] proved that, for single valued monotone operators, hemicontinuity implies demicontinuity whenever $\mathcal{D}(A)$ is a quasi dense subset of a Banach space and A is locally bounded at each point of $\mathcal{D}(A)$. So, Corollary 20 extends this result for locally convex Hausdorff spaces, showing that the singled-valuedness is a consequence of the hypothesis.

5. Closedness

An operator $A : X \to 2^{X^*}$, defined on a normed space X, is said to be *demiclosed* if for each pair of sequences (z_n) in X, converging weakly to $z \in X$, and (z_n^*) in X^*, converging in the norm topology to $z^* \in X^*$, with $z_n^* \in Az_n$, one has $z^* \in Az$.

The following propositions are extensions of well-known results for maximal monotone operators in Hilbert spaces to more general spaces. The proofs are essentially the same (see propositions 1.6 y 1.7 in [24]).

Proposition 21. *Let X be a normed space and let $A : X \to 2^{X^*}$ be a maximal monotone operator. Then A is demiclosed.*

Proof. Let (z_n) be a sequence in X that converges weakly to z while (z_n^*) is any sequence in X^* that converges strongly to z^* with $z_n^* \in Az_n$. Then, by monotonicity, we have

$$\langle x^* - z_n^*, x - z_n \rangle \geq 0 \quad \text{for all} \;\; x^* \in Ax$$

and all $x \in \mathcal{D}(A)$. Passing to the limit we obtain

$$\langle x^* - z^*, x - z \rangle \geq 0, \quad \text{for all} \;\; u^* \in Ax \;\; \text{and all} \;\; x \in \mathcal{D}(A).$$

The maximality of A gives $z \in \mathcal{D}(A)$ and $z^* \in Az$, which completes the proof. □

Remark 22. In the previous proof observe that if X is a Banach space, (z_n) converges strongly to z, and (z_n^*) (with $z_n^* \in Az_n$) converges in the weak* topology to z^*, then we also obtain that $z^* \in Az$.

Proposition 23. *Let X be a topological vector space and let $A : X \to 2^{X^*}$ be a $\mathcal{D}$-maximal monotone operator. Then, for each $x \in \mathcal{D}(A)$, Ax is a w^*-closed convex subset of X^**

Proof. We first show that Ax is convex for an arbitrary $x \in \mathcal{D}(A)$. Let $x_1^*, x_2^* \in Ax$. Then, we know that

$$\langle x_1^* - z^*, x - z\rangle \geq 0 \quad \text{and} \quad \langle x_2^* - z^*, x - z\rangle \geq 0 \quad \text{for } z \in \mathcal{D}(A) \quad \text{and} \quad z^* \in Az.$$

Let $x_t^* = tx_1^* + (1-t)x_2^*$ for $t \in [0,1]$. Then

$$\langle x_t^* - z^*, x - z\rangle = t\langle x_1^* - z^*, x - z\rangle + (1-t)\langle x_2^* - z^*, x - z\rangle \geq 0.$$

Hence $\langle x_t^* - z^*, x - z\rangle \geq 0$ for all $z \in \mathcal{D}(A)$ and $z^* \in Az$, and since A is $\mathcal{D}$-maximal monotone, we conclude that $x_t^* \in Ax$. To see that Ax is w^*-closed, let $\{x_\alpha^*\}$ be a net in Ax such that $x_\alpha^* \to x^*$. Let $z \in \mathcal{D}(A)$ and $z^* \in Az$. Then

$$\begin{aligned} \langle x^* - z^*, x - z\rangle &= \langle x^* - x_\alpha^*, x - z\rangle + \langle x_\alpha^* - z^*, x - z\rangle \\ &\geq \langle x^* - x_\alpha^*, x - z\rangle. \end{aligned}$$

Since $x_\alpha^* \to x^*$, we can deduce $\langle x^* - z^*, x - z\rangle \geq 0$. Hence $x^* \in Ax$ due to the $\mathcal{D}$-maximal monotonicity of A. □

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