

# Evolution equations for maximal monotone operators: asymptotic analysis in continuous and discrete time

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## Abstract

This survey is devoted to the asymptotic behavior of solutions of evolution equations generated by maximal monotone operators in Hilbert spaces. The emphasis is in the comparison of continuous time trajectories to sequences generated by implicit or explicit discrete time schemes. The analysis covers weak convergence for the average process, for the process itself and strong convergence. The aim is to highlight the main ideas and unifying the proofs. Furthermore the connection is made with the analysis in terms of almost orbits that allows for a broader scope.

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## Introduction

Discrete and continuous dynamical systems governed by maximal monotone operators have a great number of applications in optimization, equilibrium, fixed-point theory, partial differential equations, among others.

We are specially concerned about the connection between continuous time and discrete time models. This connection occurs at two levels:

1. On a compact interval, one approximates continuous-time trajectories by interpolation of some sequences computed via discretization. By considering vanishing step sizes this construction is used to prove existence results and to approximate the trajectories numerically.
2. Another approximation is in the long term, where we compare asymptotic properties of a continuous trajectory to similar asymptotic properties of a given path defined inductively through a sequence of values and step sizes.

It is important to mention that some estimations (e.g. Kobayashi type) can be useful for both purposes.

The literature on this subject is huge but lot of the arguments turn out to be pretty much the same. Therefore, we intend to give a concise yet complete compendium of the results available, with an emphasis on the techniques and the way they enter in the proofs.

Most of the properties will be established in the framework of Hilbert spaces since our aim is to underline unity in terms of tools and approach. A lot of results can be extended but, in most cases, additional specific assumptions are needed. With no aim for completeness, we have included several references to the corresponding results in Banach spaces that we think might be useful.

The paper is organized as follows: In section 1 we recall the basic properties of maximal monotone operators along with some examples. Section 2 deals with the associated dynamic approach. We present the existence results for the differential inclusion  $\dot{u} \in -Au$  and global properties of implicit and explicit discretizations. Section 3 establishes the convergence of the value  $f(u)$  in the case of an operator of the form  $A = \partial f$ . In section 4 we describe general results on weak convergence: tools, arguments, characterization of the weak limits. Section 5 is devoted to weak convergence in average and section 6 is concerned with weak convergence, especially for demipositive operators. In section 7 we

present the, mostly geometric, conditions ensuring that the convergence is strong. Section 8 deals with asymptotic equivalence and explains some apparently hidden relationships between certain continuous- and discrete-time dynamical systems. Finally, section 9 contains some concluding remarks.

# 1 Preliminaries

The purpose of this section is to introduce notations and to recall basic results.

## 1.1 Monotone operators

Let  $H$  be a real Hilbert space with inner product  $\langle \cdot, \cdot \rangle$  and norm  $\|\cdot\|$ . An *operator* is a set-valued mapping  $A : H \rightrightarrows H$  whose domain

$$D(A) = \{u \in H : Au \neq \emptyset\}$$

is nonempty. For convenience of notation, sometimes we will identify  $A$  with its graph by writing  $[u, u^*] \in A$  for  $u^* \in Au$ . The operator  $A^{-1}$  is defined by its graph:  $[u, u^*] \in A^{-1}$  if, and only if,  $[u^*, u] \in A$ . An operator  $A : H \rightrightarrows H$  is *monotone* if one has

$$\langle x^* - y^*, x - y \rangle \geq 0 \tag{1}$$

for all  $[x, x^*], [y, y^*] \in A$ .

A monotone operator is *maximal* if its graph is not properly contained in the graph of any other monotone operator. Observe that if  $A$  is monotone (resp. maximal monotone) then so are  $A^{-1}$  and  $\lambda A$  if  $\lambda > 0$ .

**Lemma 1** *Let  $A$  be a maximal monotone operator. A point  $[x, x^*] \in H \times H$  belongs to the graph of  $A$  if, and only if,*

$$\langle x^* - u^*, x - u \rangle \geq 0 \quad \text{for all } [u, u^*] \in A.$$

**Proof.** If  $[x, x^*] \in A$  the inequality holds by monotonicity. Conversely, if  $[x, x^*] \notin A$ , then the set  $A \cup \{[x, x^*]\}$  is the graph of a monotone operator that extends  $A$ , which contradicts maximality. ■

An operator  $A : H \rightrightarrows H$  is *nonexpansive* if one has

$$\|x^* - y^*\| \leq \|x - y\| \tag{2}$$

for all  $[x, x^*], [y, y^*] \in A$ . Observe that a nonexpansive operator is single-valued on its domain.

Let  $I$  be the identity mapping on  $H$ . For  $\lambda > 0$ , the *resolvent* of  $A$  is the operator

$$J_\lambda^A = (I + \lambda A)^{-1}.$$

**Theorem 2** *Let  $A : H \rightrightarrows H$ . Then*

- i)  $A$  is monotone if, and only if,  $J_\lambda^A$  is nonexpansive for each  $\lambda > 0$ .*
- ii) A monotone operator  $A$  is maximal if, and only if,  $I + \lambda A$  is surjective for each  $\lambda > 0$ .*

**Proof.**

*i)* Let  $A$  be monotone,  $[x, x^*], [y, y^*] \in A$  and  $\lambda > 0$ .

Inequality (1) implies

$$\|x - y\| \leq \|x - y + \lambda(x^* - y^*)\|, \quad \forall \lambda \geq 0 \tag{3}$$

which is the non expansiveness of  $J_\lambda^A$ .

Conversely, (3) leads to

$$2\lambda \langle x^* - y^*, x - y \rangle + \lambda^2 \|x^* - y^*\|^2 \geq 0$$

hence implies (1) by dividing by  $\lambda$  and letting  $\lambda \rightarrow 0$ .

*ii)* It is enough to prove the result for  $\lambda = 1$ . Given  $z_0 \in H$ , we will find  $x_0 \in H$  such that  $\langle x^* - (z_0 - x_0), x - x_0 \rangle \geq 0$  for all  $[x, x^*] \in A$  so that maximality of  $A$  implies  $z_0 - x_0 \in Ax_0$ . For  $[x, x^*] \in A$ , define the weakly compact set  $C_{x, x^*}$  by

$$C_{x, x^*} = \{x_0 \in H : \langle x^* + x_0 - z_0, x - x_0 \rangle \geq 0\}.$$

It suffices to show that the family  $\{C_{x,x^*}\}_{[x,x^*] \in A}$  has the finite intersection property. To this end take  $[x_i, x_i^*] \in A$  for  $i = 1, \dots, n$ . Let  $\Delta = \{(\lambda_1, \dots, \lambda_n) : \lambda_i \geq 0; \sum_{i=1}^n \lambda_i = 1\}$  denote the  $n$ -dimensional simplex and consider the function  $f : \Delta \times \Delta \rightarrow \mathbf{R}$  given by

$$f(\lambda, \mu) = \sum_{i=1}^n \mu_i \langle x_i^* + x(\lambda) - z_0, x(\lambda) - x_i \rangle$$

with  $x(\lambda) = \sum_{i=1}^n \lambda_i x_i$ . Clearly  $f(\cdot, \mu)$  is convex and continuous while  $f(\lambda, \cdot)$  is linear. The Min-Max Theorem (see, for instance, Theorem 1.1 in [19, Brézis]) implies the existence of  $\lambda_0 \in \Delta$  such that

$$\max_{\mu \in \Delta} f(\lambda_0, \mu) = \max_{\mu \in \Delta} \min_{\lambda \in \Delta} f(\lambda, \mu) \leq \max_{\mu \in \Delta} f(\mu, \mu).$$

Now monotonicity of  $A$  implies

$$\begin{aligned} f(\mu, \mu) &= \sum_{i=1}^n \mu_i \langle x_i^*, x(\mu) - x_i \rangle + \langle x(\mu) - z_0, x(\mu) - x(\mu) \rangle \\ &= \sum_{i,j=1}^n \mu_i \mu_j \langle x_i^*, x_j - x_i \rangle \\ &= \frac{1}{2} \sum_{i,j=1}^n \mu_i \mu_j \langle x_i^* - x_j^*, x_j - x_i \rangle \leq 0 \end{aligned}$$

so that  $f(\lambda_0, \mu) \leq 0$  for all  $\mu \in \Delta$ . Taking for  $\mu$  the extreme points we get

$$\langle y_i + x(\lambda_0) - z_0, x(\lambda_0) - x_i \rangle \leq 0$$

for all  $i$ , which is  $x(\lambda_0) \in \bigcap_{i=1}^n C_{x_i, x_i^*}$ .

Conversely, take  $[u, u^*] \in H \times H$  such that  $\langle u^* - v^*, u - v \rangle \geq 0$  for all  $[v, v^*] \in A$ . Since  $I + A$  is surjective, there is  $[\bar{v}, \bar{v}^*] \in A$  such that  $\bar{v} + \bar{v}^* = u + u^*$ . Then  $\langle u^* - \bar{v}^*, u - \bar{v} \rangle = -\|u - \bar{v}\|^2 \geq 0$  which implies  $u = \bar{v}$ ,  $u^* = \bar{v}^*$  and  $[u, u^*] \in A$ . ■

### Comments

The study of monotone operators started in [47, Minty]. See also [37, Kato] for part *i*) in Banach spaces. The *if* part in *ii*) holds in Banach spaces, essentially by the same arguments. The proof presented above for the *only if* part can be found in [19, Brézis]. This result does not hold in general Banach spaces (see [36, Hirsch]). □

## 1.2 Examples and properties

**Example 1** Let  $\Gamma_0(H)$  denote the set of all proper, lower-semicontinuous convex functions  $f : H \rightarrow \mathbf{R} \cup \{+\infty\}$ . For  $f \in \Gamma_0(H)$ , the *subdifferential* of  $f$  is the operator  $\partial f : H \rightrightarrows H$  defined by

$$\partial f(x) = \{x^* \in H : f(z) \geq f(x) + \langle x^*, z - x \rangle \text{ for all } z \in H\}.$$

To see that it is monotone, take  $x^* \in \partial f(x)$  and  $y^* \in \partial f(y)$ . Thus

$$\begin{aligned} f(y) &\geq f(x) + \langle x^*, y - x \rangle \\ f(x) &\geq f(y) + \langle y^*, x - y \rangle \end{aligned}$$

and adding these two inequalities we obtain  $\langle x^* - y^*, x - y \rangle \geq 0$ .

For maximality, according to Theorem 2 it suffices to prove that for each  $y \in H$  and each  $\lambda > 0$  there is  $x_\lambda \in D(\partial f)$  such that  $y \in x_\lambda + \lambda \partial f(x_\lambda)$ . Indeed, consider the *Moreau-Yosida approximation* of  $f$  at  $y$ , which is the function  $f_\lambda$  defined by

$$f_\lambda(x) = f(x) + \frac{1}{2\lambda} \|x - y\|^2. \quad (4)$$

It is proper, lower-semicontinuous, strongly convex and coercive (due to the quadratic term and the fact that  $f$  has a affine minorant). Its unique minimizer  $x_\lambda$  satisfies

$$0 \in \partial f_\lambda(x_\lambda) = \partial f(x_\lambda) + \frac{1}{\lambda}(x_\lambda - y).$$

That is,  $y \in x_\lambda + \lambda \partial f(x_\lambda)$ . □

**Example 2** Let  $A$  be monotone, single-valued and continuous on  $D(A) = H$ . Then  $A$  is maximal. Indeed, from  $\langle u - Ay, x - y \rangle \geq 0$  for all  $y \in H$  one deduces, with  $y = x - tw$ , that  $\langle u - A(x - tw), w \rangle \geq 0$ , for all  $t \geq 0$  and all  $w \in H$ . By letting  $t \rightarrow 0$  we obtain  $\langle u - Ax, w \rangle \geq 0$  for all  $w \in H$ , so that  $u = Ax$ .  $\square$

**Example 3** Let  $C$  be a nonempty subset of  $H$  and let  $T : C \rightarrow H$  be nonexpansive, thus single-valued on  $C$ . The operator  $A = I - T$  is monotone because

$$\begin{aligned} \langle Ax - Ay, x - y \rangle &= \|x - y\|^2 - \langle Tx - Ty, x - y \rangle \\ &\geq \|x - y\| \left[ \|x - y\| - \|Tx - Ty\| \right] \\ &\geq 0. \end{aligned}$$

If  $C = H$  maximality is given in Example 2. Otherwise,  $T$  can be extended to a nonexpansive function defined on all of  $H$ , so that  $A$  is not maximal. If  $C$  is closed and convex this extension is easily constructed by considering  $\tilde{T} = T \circ P_C$ , where  $P_C$  denotes the orthogonal projection onto  $C$ . Notice that if  $T : C \rightarrow C$  then  $\tilde{T}$  has no fixed points outside of  $C$ . Pioneer works in the extension of Lipschitz functions on general sets are [46, 38, 66, 67] but the interested reader can also consult [31] for an updated survey on the topic.

It is important to point out that this lack of maximality when  $C \subsetneq H$  is not a serious drawback, as we shall see later on (see, for instance, Remark 5).  $\square$

The set of zeroes of  $A$  is

$$\mathcal{S} = A^{-1}0 = \{x \in H; 0 \in Ax\}.$$

This set is relevant in optimization and fixed-point theory:

- If  $A = I - T$ , where  $T$  is a nonexpansive mapping, then  $\mathcal{S}$  is the set of fixed points of  $T$ .
- If  $A = \partial f$ , where  $f$  is a proper lower-semicontinuous convex function then  $\mathcal{S}$  is the set of minimizers of  $f$ .

Let us describe some topological consequences of maximal monotonicity.

**Proposition 3** *Let  $A$  be maximal monotone. For each  $x \in H$ , the set  $Ax$  is closed and convex. In particular,  $\mathcal{S}$  is closed and convex.*

**Proof.** Lemma 1 implies that

$$Ax = \{x^* \in H; \langle x^* - u^*, x - u \rangle \geq 0 \quad \text{for all } [u, u^*] \in A\}$$

hence  $Ax$  is closed and convex. Since  $A^{-1}$  is maximal monotone and  $\mathcal{S} = A^{-1}0$ , the set  $\mathcal{S}$  is closed and convex.  $\blacksquare$

**Proposition 4** *Let  $A$  be a maximal monotone operator. Then  $A$  is sequentially weak-strong and strong-weak closed.*

**Proof.** Take sequences  $\{x_n\}$  and  $\{x_n^*\}$  in  $H$  such that  $[x_n, x_n^*] \in A$  for each  $n \in \mathbf{N}$  and suppose that  $x_n \rightarrow x$  and  $x_n^* \rightarrow x^*$ , as  $n \rightarrow \infty$  (consider  $A^{-1}$  for the other case). To prove that  $[x, x^*] \in A$ , recall that by monotonicity, for all  $[u, u^*] \in A$  and all  $n \in \mathbf{N}$ , we have  $\langle x_n^* - u^*, x_n - u \rangle \geq 0$ . Letting  $n \rightarrow \infty$  the convergence assumptions imply that  $\langle x^* - u^*, x - u \rangle \geq 0$  for all  $[u, u^*] \in A$ . Hence  $[x, x^*] \in A$  by Lemma 1.  $\blacksquare$

**Remark 5** If  $C \subset H$  is closed and convex,  $T : C \rightarrow C$  is nonexpansive and  $A = I - T$ , the conclusions in Propositions 3 and 4 are true, even if  $A$  is not maximal ( $C \subsetneq H$ ).  $\square$

## 2 Dynamic approach

The forthcoming sections address, among others, the issue of finding zeroes of a (maximal) monotone operator  $A$ . The strategy is the following: we shall consider some continuous and discrete dynamical systems whose trajectories may converge, in some sense and under some conditions, to points in  $\mathcal{S} = A^{-1}0$ . In this section we present these systems along with some relevant properties.

**From now on we assume that  $A$  is a maximal monotone operator.**

### 2.1 Differential inclusion

Let us take  $x \in D(A)$  and consider the following differential inclusion:

$$\begin{cases} -\dot{u}(t) \in Au(t) & \text{a.e. on } (0, \infty) \\ u(0) = x. \end{cases} \quad (5)$$

A solution of (5) is an absolutely continuous function  $u$  from  $\mathbf{R}^+$  to  $H$  satisfying these two conditions.

Observe that  $\mathcal{S}$  is precisely the set of rest points of (5).

Monotonicity implies the following dissipative property:

**Lemma 6** *Let  $u_1$  and  $u_2$  be absolutely continuous functions satisfying  $\dot{u}_i(t) \in -Au_i(t)$  almost everywhere on  $(0, T)$ . Then the function  $t \mapsto \|u_1(t) - u_2(t)\|$  is decreasing on  $(0, T)$ .*

**Proof.** For  $t \in (0, T)$  define  $\theta(t) = \frac{1}{2}\|u_1(t) - u_2(t)\|^2$ . The hypotheses give  $\dot{\theta}(t) = \langle \dot{u}_1(t) - \dot{u}_2(t), u_1(t) - u_2(t) \rangle \leq 0$  for almost every  $t$ . ■

Immediate consequences are the following:

**Corollary 7** *Let  $y \in \mathcal{S}$  and  $u$  be a solution of (5). Then  $\lim_{t \rightarrow \infty} \|u(t) - y\|$  exists.*

**Corollary 8** *There is at most one solution of (5).*

Another aspect of dissipativity is the next property:

**Proposition 9** *The speed  $\|\dot{u}(t)\|$  is decreasing.*

**Proof.** Lemma 6 implies that for any  $h > 0$  and  $s < t$

$$\|u(t+h) - u(t)\| \leq \|u(s+h) - u(s)\|.$$

We conclude by dividing by  $h$  and taking the limit as  $h \rightarrow 0$ . ■

A basic inequality is the following:

**Proposition 10** *Let  $u$  satisfy (5) and  $[v, w] \in A$ , then:*

$$\|u(t) - v\|^2 - \|u(0) - v\|^2 \leq 2 \int_0^t \langle w, v - u(s) \rangle ds. \quad (6)$$

**Proof.** Write

$$\|u(t) - v\|^2 - \|u(0) - v\|^2 = 2 \int_0^t \langle \dot{u}(s), u(s) - v \rangle ds.$$

By monotonicity, we have  $\langle \dot{u}(s), u(s) - v \rangle \leq \langle -w, u(s) - v \rangle$ , whence the result. ■

This is the idea in the definition of *integral solution* introduced in [17] (see the proof of Theorem 19).

We shall present two approaches for the existence of a solution of (5). The first one uses the *Yosida approximation* and is the best-known in the theory of optimization in Hilbert spaces. The second one uses *proximal sequences* to approximate the function  $u$ . It is popular in the field of partial differential equations since it works naturally in arbitrary Banach spaces. Since it is less known in the optimization community we present it in detail.

But before doing so, and assuming for a moment that the differential inclusion (5) *does* have a solution, observe that by Lemma 6, for each  $t \geq 0$  the mapping  $x \mapsto u(t)$  defines a non expansive function from  $D(A)$  to itself that can be continuously extended to a map  $S_t$  from  $\overline{D(A)}$  to itself. The family  $\{S_t\}_{t \geq 0}$  is the *semi-group* generated by  $A$  and satisfies:

- i)  $S_0 = I$  and  $S_t \circ S_r = S_{t+r}$ ;
- ii)  $\|S_t x - S_t y\| \leq \|x - y\|$ ;
- iii)  $\lim_{t \rightarrow 0} \|x - S_t x\| = 0$ .

Reciprocally, given a *continuous semi-group of contractions* i.e. satisfying i), ii) and iii), from a closed convex subset  $C$  to itself, there exists a *generator*, namely a maximal monotone operator  $A$  with  $C = \overline{D(A)}$  such that  $S_t x$  coincides with  $u(t)$  for  $x \in D(A)$ , see [19, Brézis].

We will use hereafter both notations  $u(t)$  and  $S_t x$ .

## 2.2 Approach through the Yosida approximation.

### 2.2.1 The Yosida approximation

Recall that the resolvent is  $J_\lambda^A$ . The *Yosida approximation* of  $A$  is the single-valued maximal monotone operator  $A_\lambda$ ,  $\lambda > 0$ , defined by

$$A_\lambda = \frac{1}{\lambda}(I - J_\lambda^A).$$

Since  $J_\lambda^A$  is nonexpansive and everywhere defined,  $A_\lambda$  is monotone (see Example 1 above) and maximal (using Lemma 1). It is also clear that  $A_\lambda$  is Lipschitz-continuous with constant  $2/\lambda$ . Observe that  $S = A^{-1}0 = A_\lambda^{-1}0$  for all  $\lambda > 0$ .

Recall that  $P_C x$  denotes the orthogonal projection of a point  $x \in H$  onto a nonempty closed convex set  $C \subset H$ . The *minimal section* of  $A$  is the operator  $A^0$  defined by  $A^0 x = P_{Ax} 0$ , which is clearly monotone but not necessarily maximal.

The following results summarize the main properties of the resolvent and the Yosida approximation. They can be found in [19, Brézis] (see also [13, Barbu] for Banach spaces).

**Proposition 11** *With the notation introduced above we have the following:*

1.  $A_\lambda x \in A J_\lambda^A x$
2.  $\|A_\lambda x\| \leq \|A^0 x\|$ ,  $\|A_\lambda x\|$  is nonincreasing in  $\lambda$  and  $\lim_{\lambda \rightarrow 0} \|A_\lambda x\| \rightarrow \|A^0 x\|$ .
3.  $\lim_{\lambda \rightarrow 0} J_\lambda^A x = x$ .
4. If  $x_\lambda \rightarrow x$  and  $A_\lambda x_\lambda$  remains bounded as  $\lambda \rightarrow 0$ , then  $x \in D(A)$ . Moreover, if  $y$  is a cluster point of  $A_\lambda x_\lambda$  as  $\lambda \rightarrow 0$ , then  $y \in Ax$ .
5.  $A^0$  characterizes  $A$  in the following sense: If  $A$  and  $B$  are maximal monotone with common domain and  $A^0 = B^0$ , then  $A = B$ .
6.  $\lim_{\lambda \rightarrow 0} A_\lambda x = A^0 x$  and  $\overline{D(A)}$ , the (strong) closure of  $D(A)$ , is convex.

### 2.2.2 The existence result

The main result is the following:

**Theorem 12** *There exists a unique absolutely continuous function  $u : [0, +\infty) \rightarrow H$  satisfying (5). Moreover,*

1.  $\dot{u} \in L^\infty(0, \infty; H)$  with  $\|\dot{u}(t)\| \leq \|A^0 x\|$  almost everywhere.
2.  $u(t) \in D(A)$  for all  $t \geq 0$  and  $\|A^0 u(t)\|$  decreases.

3.  $A^0 u(t)$  is continuous from the right and  $u(t)$  admits a right-hand derivative for all  $t \geq 0$ ; namely  $\dot{u}(t^+) = -A^0 u(t)$  (lazy behavior).

The problem of finding a trajectory satisfying (5) was first posed and studied in [41, Komura] and [30, Crandall and Pazy]. The classical proof of Theorem 12 above can be found in [19, Brézis]. The idea is to consider the differential inclusion (5) with  $A = A_\lambda$ , which has a solution  $u_\lambda$  by virtue of the Cauchy-Lipschitz-Picard Theorem. Then one proves first that, as  $\lambda \rightarrow 0$ ,  $u_\lambda$  converges uniformly on compact intervals to some  $u$ , then that  $u$  satisfies (5) for the original  $A$ . The following estimation plays a crucial role in the proof and is interesting on its own:

$$\|u_\lambda(t) - u(t)\| \leq 2\|A^0 x\|\sqrt{\lambda t}. \quad (7)$$

Finally  $u$  is proved to have the properties enumerated in Theorem 12.

*Comments*

The same method can be extended to Banach spaces  $X$  such that both  $X$  and  $X^*$  are uniformly convex (see [37, Kato]).  $\square$

## 2.3 Approach through proximal sequences.

### 2.3.1 Proximal sequences

Given  $\{\lambda_n\}$  a sequence of positive numbers or *step sizes*, a sequence  $\{x_n\}$  is *proximal* if it satisfies

$$\begin{cases} \frac{x_n - x_{n-1}}{\lambda_n} \in -Ax_n & \text{for all } n \geq 1 \\ x_0 \in H. \end{cases} \quad (8)$$

In other words,

$$x_n = (I + \lambda_n A)^{-1} x_{n-1} = J_{\lambda_n}^A x_{n-1}. \quad (9)$$

If  $A$  is maximal monotone, the existence of such a sequence follows from Theorem 2. Observe that the first inclusion in (8) can be seen as an implicit discretization of the differential inclusion (5), called also a backward scheme. The *velocity* at stage  $n$  is

$$y_n = \frac{x_n - x_{n-1}}{\lambda_n}. \quad (10)$$

*Comments*

The notion of proximal sequences and the term *proximal* were introduced in [49, Moreau] for  $f \in \Gamma_0(H)$  and  $A = \partial f$ . In that case, finding  $x_n$  corresponds to minimizing the Moreau-Yosida approximation

$$f_{\lambda_n}(x) = f(x) + \frac{1}{2\lambda_n} \|x - x_{n-1}\|^2$$

of  $f$  at  $x_{n-1}$  (see (4)).  $\square$

Monotonicity implies the following properties:

**Lemma 13** *The sequence  $\|y_n\|$  is decreasing.*

**Proof.** The inequality  $\langle y_n - y_{n-1}, x_n - x_{n-1} \rangle \leq 0$  implies  $\langle y_n - y_{n-1}, y_n \rangle \leq 0$  and therefore  $\|y_n\| \leq \|y_{n-1}\|$ .  $\blacksquare$

This is the counterpart of Proposition 9, which states that the speed of the continuous-time trajectory given by (5) decreases.

**Proposition 14** *For any  $[x, y] \in A$*

$$\|x_{n-1} - x\|^2 \geq \|x_{n-1} - x_n\|^2 + \|x_n - x\|^2 + 2\lambda_n \langle y, x_n - x \rangle. \quad (11)$$



**Proof.** Simply observe that

$$\|x_{n-1} - x\|^2 = \|x_{n-1} - x_n\|^2 + \|x_n - x\|^2 + 2\langle x_{n-1} - x_n, x_n - x \rangle \quad (12)$$

and  $\langle x_{n-1} - x_n, x_n - x \rangle \geq \langle \lambda_n y, x_n - x \rangle$  by monotonicity. ■

This is the counterpart of (6).

In particular one has:

**Lemma 15** *Let  $x \in \mathcal{S}$ . Then  $\|x_n - x\|^2 + \lambda_n^2 \|y_n\|^2 \leq \|x_{n-1} - x\|^2$ .*

An immediate consequence is the following:

**Corollary 16** *Let  $x \in \mathcal{S}$ . The sequence  $\|x_n - x\|^2$  is decreasing, thus convergent.*

Notice the similarity with Corollary 7.

### 2.3.2 Kobayashi inequality

The following inequality, from [39, Kobayashi], provides an estimation for the distance between two proximal sequences  $\{x_k\}$  and  $\{\hat{x}_l\}$ , with step sizes  $\{\lambda_k\}$  and  $\{\hat{\lambda}_l\}$ , respectively.

**We use the following notation throughout the paper:**

$$\sigma_k = \sum_{i=1}^k \lambda_i \quad \text{and} \quad \tau_k = \sum_{i=1}^k \lambda_i^2$$

(similarly for  $\hat{\sigma}_l$  and  $\hat{\tau}_l$ ).

**Proposition 17 (Kobayashi inequality)** *Let  $\{x_k\}$  and  $\{\hat{x}_l\}$  be two proximal sequences. If  $u \in D(A)$ , then*

$$\|x_k - \hat{x}_l\| \leq \|x_0 - u\| + \|\hat{x}_0 - u\| + \|A^0 u\| \sqrt{(\sigma_k - \hat{\sigma}_l)^2 + \tau_k + \hat{\tau}_l}. \quad (13)$$

We first prove the following auxiliary result:

**Lemma 18** *Let  $[u_1, v_1], [u_2, v_2] \in A$  and  $\lambda, \mu > 0$ , then*

$$(\lambda + \mu)\|u_1 - u_2\| \leq \lambda\|u_2 + \mu v_2 - u_1\| + \mu\|u_1 + \lambda v_1 - u_2\|.$$

**Proof.** Write  $\Delta u = u_1 - u_2$ . Then

$$\begin{aligned} (\lambda + \mu)\|u_1 - u_2\|^2 &= \lambda\langle u_2 - u_1, -\Delta u \rangle + \mu\langle u_1 - u_2, \Delta u \rangle \\ &= \lambda\langle u_2 + \mu v_2 - u_1, -\Delta u \rangle + \mu\langle u_1 + \lambda v_1 - u_2, \Delta u \rangle + \lambda\mu\langle v_2 - v_1, u_1 - u_2 \rangle \\ &\leq \left[ \lambda\|u_2 + \mu v_2 - u_1\| + \mu\|u_1 + \lambda v_1 - u_2\| \right] \|u_1 - u_2\| \end{aligned}$$

by monotonicity. ■

**Proof of Proposition 17:** To simplify notation set

$$c_{k,l} = \sqrt{(\sigma_k - \hat{\sigma}_l)^2 + \tau_k + \hat{\tau}_l}.$$

The proof will use induction on the pair  $(k, l)$ .

First, let us establish inequality (13) for the pair  $(k, 0)$  with  $k \geq 0$ . Monotonicity implies, using (3) that, for any  $u \in H$

$$\|x_1 - u\| \leq \|x_1 - u + \lambda_1(-y_1 - A^0 u)\| = \|x_0 - u - \lambda_1 A^0 u\|$$

so that

$$\|x_1 - u\| \leq \|x_0 - u\| + \lambda_1 \|A^0 u\|.$$

Inductively we obtain

$$\|x_k - u\| \leq \|x_0 - u\| + \sigma_k \|A^0 u\|.$$

Thus

$$\begin{aligned} \|x_k - \widehat{x}_0\| &\leq \|x_k - u\| + \|u - \widehat{x}_0\| \\ &\leq \|x_0 - u\| + \sigma_k \|A^0 u\| + \|\widehat{x}_0 - u\| \\ &\leq \|x_0 - u\| + \|\widehat{x}_0 - u\| + c_{k,0} \|A^0 u\| \end{aligned}$$

because  $\sigma_k \leq c_{k,0}$ . In a similar fashion we prove the inequality for  $(0, l)$  with  $l \geq 0$ .

Now suppose (13) holds for  $(k-1, l)$  and  $(k, l-1)$ . According to Lemma 18,

$$(\lambda_k + \widehat{\lambda}_l) \|x_k - \widehat{x}_l\| \leq \lambda_k \|\widehat{x}_l + \widehat{\lambda}_l \widehat{y}_l - x_k\| + \widehat{\lambda}_l \|x_k + \lambda_k y_k - \widehat{x}_l\|.$$

Setting  $\alpha_{k,l} = \frac{\widehat{\lambda}_l}{\lambda_k + \widehat{\lambda}_l}$  and  $\beta_{k,l} = 1 - \alpha_{k,l} = \frac{\lambda_k}{\lambda_k + \widehat{\lambda}_l}$  we have

$$\begin{aligned} \|x_k - \widehat{x}_l\| &\leq \alpha_{k,l} \|x_{k-1} - \widehat{x}_l\| + \beta_{k,l} \|\widehat{x}_{l-1} - x_k\| \\ &\leq \alpha_{k,l} [\|x_0 - u\| + \|\widehat{x}_0 - u\| + c_{k-1,l} \|A^0 u\|] \\ &\quad + \beta_{k,l} [\|x_0 - u\| + \|\widehat{x}_0 - u\| + c_{k,l-1} \|A^0 u\|] \\ &= \|x_0 - u\| + \|\widehat{x}_0 - u\| + [\alpha_{k,l} c_{k-1,l} + \beta_{k,l} c_{k,l-1}] \|A^0 u\|. \end{aligned} \tag{14}$$

It only remains to verify that

$$\alpha_{k,l} c_{k-1,l} + \beta_{k,l} c_{k,l-1} \leq c_{k,l}. \tag{15}$$

Cauchy-Schwartz Inequality implies

$$\begin{aligned} \alpha_{k,l} c_{k-1,l} + \beta_{k,l} c_{k,l-1} &= \alpha_{k,l}^{1/2} (\alpha_{k,l}^{1/2} c_{k-1,l}) + \beta_{k,l}^{1/2} (\beta_{k,l}^{1/2} c_{k,l-1}) \\ &\leq (\alpha_{k,l} + \beta_{k,l})^{1/2} (\alpha_{k,l} c_{k-1,l}^2 + \beta_{k,l} c_{k,l-1}^2)^{1/2} \\ &= (\alpha_{k,l} c_{k-1,l}^2 + \beta_{k,l} c_{k,l-1}^2)^{1/2}. \end{aligned}$$

On the other hand, notice that  $c_{k-1,l}^2 = c_{k,l}^2 - 2\lambda_k(\sigma_k - \widehat{\sigma}_l)$ , while  $c_{k,l-1}^2 = c_{k,l}^2 + 2\widehat{\lambda}_l(\sigma_k - \widehat{\sigma}_l)$ . Hence,

$$\begin{aligned} (\alpha_{k,l} c_{k-1,l} + \beta_{k,l} c_{k,l-1})^2 &\leq \alpha_{k,l} c_{k-1,l}^2 + \beta_{k,l} c_{k,l-1}^2 \\ &= \alpha_{k,l} c_{k,l}^2 + \beta_{k,l} c_{k,l}^2 - 2(\alpha_{k,l} \lambda_k - \beta_{k,l} \widehat{\lambda}_l)(\sigma_k - \widehat{\sigma}_l) \\ &= c_{k,l}^2. \end{aligned}$$

Inequalities (14) and (15) give (13). ■

### Comments

Kobayashi's original inequality also accounts for possible errors in the determination of the proximal sequence, see [39, Kobayashi]. Nonautonomous versions of the inequality can be found in [40, Kobayasi, Kobayashi and Oharu] or [2, Alvarez and Peyrouquet]. □

### 2.3.3 The existence result

In general Banach spaces, existence and uniqueness of a solution of (5) can also be derived by the following method from [29, Crandall and Liggett] based on the resolvent.

Set  $t \in [0, T]$ ,  $m \in \mathbf{N}$  and consider a proximal sequence with constant step sizes  $\lambda_k \equiv t/m$ . The  $m$ -th iteration defines a function

$$u_m(t) = \left( I + \frac{t}{m} A \right)^{-m} x.$$

Repeat the procedure for each  $m$  to obtain a sequence  $\{u_m(t)\}$  of functions from  $[0, T]$  to  $H$ .

**Theorem 19** *The sequence  $\{u_m(t)\}$  defined above converges to some  $u(t)$  uniformly on every compact interval  $[0, T]$ . Moreover, the function  $t \mapsto u(t)$  satisfies (5).*

**Proof.** Instead of the original proof from [29, Crandall and Liggett] we present an easier one using Kobayashi's inequality (13)<sup>1</sup>. Fix  $N, M \in \mathbf{N}$  and  $t, s \in [0, T]$  with  $T > 0$ . Consider two proximal sequences with  $\lambda_k = t/N$  and  $\hat{\lambda}_l = s/M$  for all  $k, l$ . Initialize  $x_k$  and  $\hat{x}_l$  both at  $x$ . Note that  $x_N = u_N(t)$  and  $\hat{x}_M = u_M(s)$  hence

$$\|u_N(t) - u_M(s)\| \leq \|A^0 x\| \sqrt{(t-s)^2 + \frac{T^2}{N} + \frac{T^2}{M}}.$$

Thus the sequence  $\{u_n\}$  converges uniformly on  $[0, T]$  to a function  $u$ , which is globally Lipschitz-continuous with constant  $\|A^0 x\|$ .

In order to prove that the function  $u$  satisfies (5) it suffices to verify that it is an *integral solution* in the sense of [17, B enilan] (see Proposition 10), which means that for all  $[x, y] \in A$  and  $t > s \geq 0$  we have

$$\frac{1}{2} [\|u(t) - x\|^2 - \|u(s) - x\|^2] \leq \int_s^t \langle y, x - u(\tau) \rangle d\tau. \quad (16)$$

Since  $u$  is absolutely continuous and  $A$  is maximal monotone, (16) implies  $\dot{u}(t) \in -Au(t)$  almost everywhere on  $[0, T]$ . Monotonicity of  $A$  implies that for any proximal sequence  $\{x_k\}$ : one has  $\langle x_{k-1} - x_k - \lambda_k y, x_k - x \rangle \geq 0$ . But  $\|x_k - x\|^2 - \|x_{k-1} - x\|^2 \leq 2\langle x_{k-1} - x_k, x - x_k \rangle$  and so

$$\|x_k - x\|^2 - \|x_{k-1} - x\|^2 \leq 2\lambda_k \langle y, x - x_k \rangle.$$

Summing up for  $k = 1, \dots, n$  we obtain

$$\|x_n - x\|^2 - \|x_0 - x\|^2 \leq 2 \sum_{k=1}^n \lambda_k \langle y, x - x_k \rangle.$$

Setting  $x_0 = u(s)$  and passing to the limit appropriately we get (16). Notice that  $u(t) \in D(A)$  by maximality. ■

A consequence of Proposition 17 and Theorem 19 is the following:

**Corollary 20** *The following statements hold:*

i) *For each  $z \in D(A)$  we have*

$$\|x_n - u(t)\| \leq \|x_0 - z\| + \|u(0) - z\| + \|A^0 z\| \sqrt{(\sigma_n - t)^2 + \tau_n}.$$

ii) *For trajectories  $u$  and  $v$  we get*

$$\|v(s) - u(t)\| \leq \|v(0) - z\| + \|u(0) - z\| + \|A^0 z\| |s - t|.$$

iii) *The unique function  $u$  satisfying (5) is Lipschitz-continuous with*

$$\|u(s) - u(t)\| \leq \|A^0 u(0)\| |s - t|.$$

iv)  *$\dot{u} \in L^\infty(0, \infty; H)$  with  $\|\dot{u}(t)\| \leq \|A^0 x\|$  almost everywhere.*

Proposition 17 was used to construct a continuous trajectory by considering finer and finer discretizations on a compact interval. By controlling the distance between two discrete schemes it is possible to obtain bounds for the distance between a limit trajectory and a discrete scheme. As a consequence, one can estimate the distance between two trajectories as well.

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<sup>1</sup>In fact, Kobayashi's proof is based on a simplification of Crandall and Liggett's method.

## 2.4 Euler sequences

Assume  $A$  maps  $D(A)$  into itself (this is a strong assumption, so the range of applications of this discretization method is limited compared to proximal sequences). Let  $\{\lambda_n\}$  be a sequence of numbers in  $(0, 1]$  (the step sizes). Define an *Euler sequence*  $\{z_n\}$  recursively by

$$\begin{cases} \frac{z_n - z_{n-1}}{\lambda_{n-1}} \in -Az_{n-1} & \text{for all } n \geq 1 \\ z_0 \in D(A). \end{cases} \quad (17)$$

A remarkable feature of this scheme is that the terms of the sequence can be computed explicitly (forward scheme).

Observe that if  $A = I - T$  with  $T : C \rightarrow C$  nonexpansive and  $\lambda_n \equiv 1$  then  $z_n = T^n z_0$ . This particular case has been studied extensively by several authors in the search for fixed points of  $T$ . Some of their results will be presented in the forthcoming sections.

Notice also that in this framework,  $A = I - T$  with  $T$  nonexpansive, a Kobayashi-type inequality holds too, namely

$$\|z_k - \widehat{z}_l\| \leq \|z_0 - u\| + \|\widehat{z}_0 - u\| + \|u - T(u)\| \sqrt{(\sigma_k - \widehat{\sigma}_l)^2 + \tau_k + \widehat{\tau}_l}, \quad (18)$$

where  $u$  is any point in  $H$ . This fact was recently established by [68, Vigerál].

Let us define the velocity at stage  $n$  as

$$w_n = \frac{z_{n+1} - z_n}{\lambda_n} \in -Az_n. \quad (19)$$

**Lemma 21** *If  $[u, v] \in A$  then*

$$\|z_{n+1} - u\|^2 \leq \|z_n - u\|^2 + 2\lambda_n \langle v, u - z_n \rangle + \lambda_n^2 \|w_n\|^2. \quad (20)$$

**Proof.** For any  $u \in H$  one has

$$\|z_{n+1} - u\|^2 = \|z_n - u\|^2 + 2\lambda_n \langle w_n, z_n - u \rangle + \lambda_n^2 \|w_n\|^2. \quad (21)$$

The desired inequality follows from monotonicity since  $\langle w_n, z_n - u \rangle \leq \langle v, u - z_n \rangle$  for  $[u, v] \in A$ . ■

This is the counterpart of (6) and (11). In particular one has:

**Lemma 22** *If  $u \in \mathcal{S}$  then  $\|z_{n+1} - u\|^2 \leq \|z_n - u\|^2 + \lambda_n^2 \|w_n\|^2$ .*

Observe the similarity and the difference with (5) and (8). The dissipativity condition in Lemma 22 is much weaker than the corresponding ones in Lemmas 6 and 15.

An immediate consequence is the following:

**Corollary 23** *Assume  $\sum \|z_{n+1} - z_n\|^2 < \infty$ . For each  $u \in \mathcal{S}$  the sequence  $\|z_n - u\|$  is convergent.*

**Proof.** It suffices to observe from Lemma 22 that the sequence  $\|z_n - u\|^2 + \sum_{m=n}^{+\infty} \|z_{m+1} - z_m\|^2$  is decreasing. ■

*Comments*

The hypothesis in the previous result holds if  $\{\lambda_n\} \in \ell^2$  and  $\{w_n\}$  is bounded. □

Notice the similarity with Corollaries 7 and 16.

The main drawback of Euler sequences is that they can be quite unstable. Most convergence results need regularity assumptions such as  $\{\lambda_n\} \in \ell^2$  and the boundedness of the sequence  $\{w_n\}$ , or at least that  $\sum \|z_{n+1} - z_n\|^2 < \infty$ .

An important result involving an operator  $A$  of the form  $I - T$  is the following, see [19, Brézis]:

**Proposition 24 (Chernoff's estimate)** *Let  $T$  be nonexpansive from  $H$  to itself and  $\lambda > 0$ . If  $v$  satisfies*

$$\dot{v}(t) = -\frac{1}{\lambda}(I - T)v(t)$$

with  $v(0) = v_0$  then

$$\|v(t) - T^n v_0\| \leq \|\dot{v}(0)\| \sqrt{\lambda t + (n\lambda - t)^2}. \quad (22)$$

**Proof.** It is enough to consider the case  $\lambda = 1$ .

Define  $\phi_n(t) = \|v(t) - T^n v_0\|$  and  $\gamma_n(t) = \|\dot{v}(0)\| \sqrt{t + [n - t]^2}$ . We shall prove inductively that  $\phi_n(t) \leq \gamma_n(t)$ . For  $n = 0$  simply observe that

$$\|v(t) - v_0\| \leq \int_0^t \|\dot{v}(s)\| ds \leq \|\dot{v}(0)\| t \leq \gamma_0(t)$$

by Proposition 9.

Now let us assume  $\phi_{n-1} \leq \gamma_{n-1}$  and prove  $\phi_n \leq \gamma_n$ . Multiplying  $\dot{v}(t) + v(t) = Tv(t)$  by  $e^t$  and integrating we obtain  $v(t) = v_0 e^{-t} + \int_0^t e^{(s-t)} T v(s) ds$  so that

$$\begin{aligned} \phi_n(t) &= \left\| e^{-t}(v_0 - T^n v_0) + \int_0^t e^{(s-t)} [T v(s) - T^n v_0] ds \right\| \\ &\leq e^{-t} \|v_0 - T^n v_0\| + \int_0^t e^{(s-t)} \phi_{n-1}(s) ds. \end{aligned}$$

Noticing that  $\|v_0 - T^n v_0\| \leq \sum_{i=1}^n \|T^{i-1} v_0 - T^i v_0\| \leq n \|v_0 - T v_0\| = n \|\dot{v}(0)\|$  and using the induction hypothesis we deduce

$$\phi_n(t) \leq e^{-t} \left[ n \|\dot{v}(0)\| + \int_0^t e^s \gamma_{n-1}(s) ds \right].$$

Hence it suffices to establish the inequality

$$n + \int_0^t e^s \sqrt{s + [(n-1) - s]^2} ds \leq e^t \sqrt{t + [n - t]^2}.$$

Since this holds trivially for  $t = 0$ , it suffices to prove the inequality for the derivatives

$$e^t \sqrt{t + [(n-1) - t]^2} \leq e^t \left[ \sqrt{t + [n - t]^2} + \frac{1 - 2[n - t]}{2\sqrt{t + [n - t]^2}} \right].$$

This is easily verified by squaring both sides. ■

In particular, setting  $T = J_\lambda^A$  we get  $v = u_\lambda$  as in (7). Combining inequalities (7) and (22) we deduce that

$$\begin{aligned} \|(I + \lambda A)^{-n} x - u(t)\| &\leq \|(J_\lambda^A)^n x - u_\lambda(t)\| + \|u_\lambda(t) - u(t)\| \\ &\leq \|A^0 x\| \left( 2\sqrt{\lambda t} + \sqrt{\lambda t + (n\lambda - t)^2} \right). \end{aligned} \quad (23)$$

Taking  $\lambda = t/n$  we obtain the following exponential approximation

$$\left\| \left( I + \frac{t}{n} A \right)^{-n} x - u(t) \right\| \leq \frac{3 \|A^0 x\| t}{\sqrt{n}}. \quad (24)$$

Therefore, this discretization also approximates the continuous-time trajectory. Moreover, the approximation is uniform on bounded intervals.

## 2.5 Further remarks

### 2.5.1 Discrete to continuous

Given a sequence  $\{x_n\}$  in  $X$  along with a strictly increasing sequence  $\{\sigma_n\}$  of positive numbers with  $\sigma_0 = 0$  and  $\sigma_n \rightarrow \infty$  as  $n \rightarrow \infty$ , one can construct a “continuous-time” trajectory  $x$  by interpolation: for  $t \in [\sigma_n, \sigma_{n+1}]$ , take  $x(t)$  anywhere on the segment  $[x_n, x_{n+1}]$ . It is easy to see that any trajectory defined this way converges to some  $\bar{x}$  if, and only if, the sequence  $\{x_n\}$  converges to  $\bar{x}$ .

Observe that if the interpolation is chosen to be piecewise constant in each subinterval  $[\sigma_n, \sigma_{n+1})$ , then

$$\frac{1}{t} \int_0^t x(\xi) d\xi = \frac{1}{\sigma_n} \sum_{k=1}^n \lambda_k x_k,$$

where  $\lambda_k = \sigma_k - \sigma_{k-1}$ . The sum on the right-hand side of the previous equality represents an average of the points  $\{x_n\}$  that is *weighted* by the sequence  $\{\lambda_n\}$  and will be denoted by  $\bar{x}_n$ . Observe also that the convergence of these weighted averages is equivalent to the convergence of the continuous-time interpolation.

**From now on we will consider only proximal or Euler sequences with step sizes  $\{\lambda_n\} \notin \ell^1$ .**

### 2.5.2 Asymptotic analysis to be carried out in the following sections

The next sections are devoted to the asymptotic analysis. We start by considering the sequences of values in the case  $f \in \Gamma_0(H)$  and  $A = \partial f$  in Section 3. The rest deals with the behavior of trajectories and sequences themselves. Section 4 presents general tools related to weak convergence and properties of weak limit points. These last properties hold under weaker assumptions for the averages, which are studied in Section 5. In Section 6 we present weak convergence, in particular in the framework of demipositive operators. Section 7 introduces different geometrical conditions that are sufficient for strong convergence. Section 8 is devoted to almost orbits and describes equivalence classes that allow to recover previous results with a new perspective and extend to non autonomous processes.

## 3 Convex optimization and convergence of the values

This section is devoted to the case where  $A = \partial f$  is the subdifferential of a proper lower-semicontinuous convex function. We evaluate  $f$  on trajectories and discuss on the behavior of its values.

### 3.1 Continuous dynamics

When  $A = \partial f$  with  $f \in \Gamma_0(H)$ , the differential inclusion (5) is a generalization of the gradient method, for nondifferentiable functions. In what follows let  $u : [0, \infty) \rightarrow H$  be the solution of the differential inclusion

$$\dot{u}(t) \in -\partial f(u(t)), \tag{25}$$

whose existence is given in Theorem 12. Let

$$f^* = \inf_{x \in H} f(x) \in \mathbf{R} \cup \{-\infty\}.$$

The following result and its proof are essentially from [19, Brézis] (see [34, Güler]).

**Proposition 25** *The function  $t \mapsto f(u(t))$  is decreasing and  $\lim_{t \rightarrow \infty} f(u(t)) = f^*$ .*

**Proof.** The subdifferential inequality is

$$f(u(t)) - f(u(s)) \leq -\langle \dot{u}(t), u(t) - u(s) \rangle.$$

Thus

$$\limsup_{s \rightarrow t^-} \frac{f(u(t)) - f(u(s))}{t - s} \leq -\|\dot{u}(t)\|^2$$

and so the function  $t \mapsto f(u(t))$  is decreasing.

For each  $z \in H$  and  $s \in [0, t]$  the subdifferential inequality then gives

$$f(z) \geq f(u(s)) + \langle \dot{u}(s), u(s) - z \rangle \geq f(u(t)) + \frac{1}{2} \frac{d}{ds} \|u(s) - z\|^2.$$

Integrating on  $[0, t]$  we obtain that

$$tf(z) \geq tf(u(t)) + \frac{1}{2} \|u(t) - z\|^2 - \frac{1}{2} \|u(0) - z\|^2$$

and so

$$f(u(t)) + \frac{\|u(t) - z\|^2}{2t} \leq f(z) + \frac{\|u(0) - z\|^2}{2t} \quad (26)$$

for every  $z \in H$ . We conclude by letting  $t \rightarrow \infty$ .  $\blacksquare$

### Comments

By inequality (26), if  $\mathcal{S} \neq \emptyset$  then  $f(u(t))$  converges to  $f^*$  at a rate of  $O(1/t)$ . However, if the trajectory  $u(t)$  is known to have a strong limit, then the rate drops to  $o(1/t)$  (see [34, Güler]).  $\square$

## 3.2 Proximal sequences

Let  $\{x_n\}$  be a proximal sequence associated to  $A = \partial f$ . The following result is due to [33, Güler]:

**Proposition 26** *The sequence  $f(x_n)$  is decreasing and  $\lim_{n \rightarrow \infty} f(x_n) = f^*$ .*

**Proof.** Recall that  $-y_n = -\frac{x_n - x_{n-1}}{\lambda_n} \in \partial f(x_n)$ . The subdifferential inequality implies

$$f(x_{n-1}) - f(x_n) \geq \lambda_n \|y_n\|^2 \quad (27)$$

so that  $f(x_n)$  is decreasing. Convergence of  $f(x_n)$  to  $f^*$  follows from Lemma 27 below since  $\sigma_n \rightarrow \infty$ .  $\blacksquare$

**Lemma 27** *Let  $u \in \text{dom} f$ , then*

$$f(x_n) - f(u) \leq \frac{\|u - x_0\|^2}{2\sigma_n} - \frac{\|u - x_n\|^2}{2\sigma_n} - \frac{\sigma_n}{2} \|y_n\|^2.$$

**Proof.** The subdifferential inequality gives

$$f(u) - f(x_n) \geq \langle u - x_n, -y_n \rangle = \frac{\langle u - x_n, x_{n-1} - x_n \rangle}{\lambda_n}$$

for all  $u$  in the domain of  $f$ . Thus

$$2\lambda_n(f(u) - f(x_n)) \geq \|u - x_n\|^2 + \lambda_n^2 \|y_n\|^2 - \|u - x_{n-1}\|^2.$$

Summation from 1 to  $n$  leads to

$$2\sigma_n f(u) - 2 \sum_{k=1}^n \lambda_k f(x_k) \geq \|u - x_n\|^2 + \sum_{k=1}^n \lambda_k^2 \|y_k\|^2 - \|u - x_0\|^2. \quad (28)$$

Multiplying (27) by  $\sigma_{n-1}$  and rearranging we get

$$\sigma_{n-1} f(x_{n-1}) - \sigma_n f(x_n) + \lambda_n f(x_n) \geq \lambda_n \sigma_{n-1} \|y_n\|^2,$$

from which we derive

$$-\sigma_n f(x_n) + \sum_{k=1}^n \lambda_k f(x_k) \geq \sum_{k=1}^n \lambda_k \sigma_{k-1} \|y_k\|^2$$

by summation. Adding twice this inequality to (28) we obtain

$$2\sigma_n(f(u) - f(x_n)) \geq \|u - x_n\|^2 - \|u - x_0\|^2 + \sum_{k=1}^n \lambda_k^2 \|y_k\|^2 + 2 \sum_{k=1}^n \lambda_k \sigma_{k-1} \|y_k\|^2.$$

Recall from Lemma 13 that  $\|y_n\|$  is decreasing. We get

$$\begin{aligned} \|y_n\|^2 \sigma_n^2 &= \|y_n\|^2 (\sigma_{n-1} + \lambda_n)^2 = \|y_n\|^2 (\lambda_n^2 + 2\lambda_n \sigma_{n-1} + \sigma_{n-1}^2) \\ &= \|y_n\|^2 \sum_{k=1}^n (\lambda_k^2 + 2\lambda_k \sigma_{k-1}) \leq \sum_{k=1}^n (\lambda_k^2 + 2\lambda_k \sigma_{k-1}) \|y_k\|^2 \end{aligned}$$

and the result follows at once by rearranging the terms.  $\blacksquare$

#### Comments

If  $\mathcal{S} \neq \emptyset$ , Lemma 27 gives

$$\|y_n\| \leq \frac{d(x_0, \mathcal{S})}{\sigma_n}. \quad (29)$$

A similar estimation had been proved in [20, Brézis and Lions] but the right-hand side is  $\sqrt{2}$  times larger.

The fact that  $f(x_n) \rightarrow f^*$  had first been proved in [45, Martinet] when  $f$  is coercive and  $\lambda_n \equiv \lambda$ .

By Lemma 27, if  $\mathcal{S} \neq \emptyset$  the rate of convergence of  $f(x_n)$  to  $f^*$  can be estimated at  $O(1/\sigma_n)$ . Moreover, (29) and the subdifferential inequality together give

$$f(x_n) - f^* \leq \langle x^* - x_n, -y_n \rangle \leq \|x^* - x_n\| \|y_n\| \leq \frac{d(x_0, \mathcal{S}) \|x^* - x_n\|}{\sigma_n}$$

for all  $x^* \in \mathcal{S}$ . Therefore, if the sequence  $\{x_n\}$  is known to converge strongly, then  $|f(x_n) - f^*| = o(1/\sigma_n)$ . This was proved in [33, Güler] using a clever but unnecessarily sophisticated argument instead of inequality (29).  $\square$

### 3.3 Euler sequences

Let  $\{z_n\}$  be an Euler sequence associated to  $A = \partial f$ . In this case the sequence  $f(z_n)$  need not be decreasing. However, we have the following:

**Lemma 28** *If either i)  $\sum \|z_{n+1} - z_n\|^2 < \infty$  or ii)  $\lim_{n \rightarrow \infty} \lambda_n \|w_n\|^2 = 0$ , then  $\liminf_{n \rightarrow \infty} f(z_n) = f^*$ .*

**Proof.** Assume i). Since  $-w_n \in \partial f(z_n)$ , the subdifferential inequality and (21) together imply

$$\|z_{n+1} - y\|^2 \leq \|z_n - y\|^2 + 2\lambda_n(f(y) - f(z_n)) + \lambda_n^2 \|w_n\|^2 \quad (30)$$

for each  $y \in H$ . If  $\sum \|z_{n+1} - z_n\|^2 < \infty$  then

$$\sum \lambda_n (f(z_n) - f(y)) < \infty$$

(possibly  $-\infty$ ). Since  $\{\lambda_n\} \notin \ell^1$  one must have  $\liminf_{n \rightarrow \infty} f(z_n) \leq f(y)$  for each  $y \in H$ .

Consider now ii). Inequality (30) can be rewritten as

$$\lambda_n \left[ 2(f(z_n) - f(y)) - \lambda_n \|w_n\|^2 \right] \leq \|z_n - y\|^2 - \|z_{n+1} - y\|^2$$

so that

$$\sum \lambda_n \left[ 2(f(z_n) - f(y)) - \lambda_n \|w_n\|^2 \right] < \infty$$

and  $\liminf_{n \rightarrow \infty} f(z_n) \leq f(y)$  for each  $y \in H$ .  $\blacksquare$

Part of the ideas in the proof of the preceding result (under hypothesis ii)) are from [64, Shor], where we can also find the following:



**Proposition 29** *Let  $\dim(H) < \infty$  and assume  $\mathcal{S}$  is nonempty and compact. If  $\lim_{n \rightarrow \infty} \lambda_n = 0$  and the sequence  $w_n$  is bounded then  $\lim_{n \rightarrow \infty} f(z_n) = f^*$ .*

**Proof.** By continuity, it suffices to prove that  $\text{dist}(z_n, \mathcal{S}) = \inf_{y \in \mathcal{S}} \|z_n - y\|$  tends to 0 as  $n \rightarrow \infty$ . For  $\gamma > f^*$  define  $L_\gamma = \{x : f(x) = \gamma\}$  and denote  $L_\gamma^{co}$  its convex hull. Both sets are compact. Take  $\varepsilon > 0$  and define

$$\delta(\varepsilon) = \text{dist}(\mathcal{S}, L_{f^* + \varepsilon}) \quad \text{and} \quad d(\varepsilon) = \max_{u \in L_{f^* + \varepsilon}^{co}} \text{dist}(u, \mathcal{S}).$$

Observe that  $0 < \delta(\varepsilon) \leq d(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . By hypothesis and Lemma 28 there is  $N \in \mathbf{N}$  such that  $f(z_N) \leq f^* + \varepsilon$  and  $\lambda_n \|w_n\| \leq \delta(\varepsilon)$  for all  $n \geq N$ . We shall prove that  $\text{dist}(z_n, \mathcal{S}) \leq 2d(\varepsilon)$  for all  $n \geq N$ . Since  $\varepsilon > 0$  is arbitrary this shows that  $\lim_{n \rightarrow \infty} \text{dist}(z_n, \mathcal{S}) = 0$ .

Indeed, if  $f(z_n) \leq f^* + \varepsilon$  (this holds for  $n = N$ ) then  $z_n \in L_{f^* + \varepsilon}^{co}$  and  $\text{dist}(z_n, \mathcal{S}) \leq d(\varepsilon)$ . Hence  $\text{dist}(z_{n+1}, \mathcal{S}) \leq d(\varepsilon) + \delta(\varepsilon) \leq 2d(\varepsilon)$ . On the other hand, if  $f(z_n) > f^* + \varepsilon$  then  $\text{dist}(z_{n+1}, \mathcal{S}) \leq \text{dist}(z_n, \mathcal{S})$ . To see this, notice that if  $y \in \mathcal{S}$  then  $\langle \frac{w_n}{\|w_n\|}, y - z_n \rangle$  is the distance from  $y$  to the hyperplane  $\Pi_n = \{x : \langle w_n, z_n - x \rangle = 0\}$ , which is a supporting hyperplane for the set  $L_{f(z_n)}^{co}$  at the point  $z_n$ . Therefore we have

$$\langle w_n, y - z_n \rangle \geq \|w_n\| \text{dist}(\mathcal{S}, \Pi_n) \geq \|w_n\| \text{dist}(\mathcal{S}, L_{f(z_n)}) \geq \|w_n\| \delta(\varepsilon),$$

where the second inequality follows from convexity and the last one is true whenever  $f(z_n) > f^* + \varepsilon$ . Using (21) and recalling that  $\lambda_n \|w_n\| \leq \delta(\varepsilon)$  we deduce that

$$\text{dist}(z_{n+1}, \mathcal{S})^2 \leq \text{dist}(z_n, \mathcal{S})^2 - \lambda_n \|w_n\| \delta(\varepsilon),$$

proving that  $\text{dist}(z_{n+1}, \mathcal{S}) \leq \text{dist}(z_n, \mathcal{S})$ . ■

Observe that this result does not require the stabilizing summability condition but it is necessary to make a very strong assumption on the set  $\mathcal{S}$ .

## 4 General tools for weak convergence

We denote by  $\Omega[u(t)]$  (resp.  $\Omega[x_n]$ ) the set of weak cluster points of a trajectory  $u(t)$  as  $t \rightarrow \infty$  (resp. of a sequence  $\{x_n\}$  as  $n \rightarrow \infty$ ).

Given a trajectory  $u(t)$  we define

$$\bar{u}(t) = \frac{1}{t} \int_0^t u(\xi) \, d\xi.$$

Similarly, given a sequence  $\{x_n\}$  in  $H$  along with step sizes  $\{\lambda_n\}$ , we introduce

$$\bar{x}_n = \frac{1}{\sigma_n} \sum_{k=1}^n \lambda_k x_k.$$

### 4.1 Existence of the limit

Most of the results on weak convergence that exist in the literature rely on the combination of two types of properties involving a subset  $F \subset H$  (in all that follows  $F$  will be closed and convex):

The first one is a kind of ‘‘Lyapounov condition’’ on the sequence or the trajectory like

- (a1)  $\|x_n - u\|$  converges to some  $\ell(u)$  for each  $u \in F$ , or
- (a2)  $P_F(x_n)$  converges strongly.

These properties imply that the sequence is somehow ‘‘anchored’’ to the set  $F$ .

The second one is a global one, concerning the set of weak cluster points of the sequence or trajectory:

- (b)  $\Omega[x_n] \subset F$ .

However, it is sometimes available only for the averages:

(b')  $\Omega[\bar{x}_n] \subset F$ .

The following result is a very useful tool for proving weak convergence of a sequence on the basis of (a1) and (b) above. It is known, especially in Hilbert spaces, as *Opial's Lemma* [51].

**Lemma 30 (Opial's Lemma)** *Let  $\{x_n\}$  be a sequence in  $H$  and let  $F \subset H$ . Assume*

1.  $\|x_n - u\|$  has a limit as  $n \rightarrow \infty$  for each  $u \in F$ ; and
2.  $\Omega[x_n] \subset F$ .

*Then  $x_n$  converges weakly to some  $x^* \in F$ .*

**Proof.** Since  $\{x_n\}$  is bounded it suffices to prove that it has only one weak cluster point. Let  $x, y \in \Omega[x_n] \subset F$  so that  $\|x_n - x\|$  converges to  $\ell(x)$  and similarly for  $y$ . From

$$\|x_n - y\|^2 = \|x_n - x\|^2 + \|x - y\|^2 + 2\langle x_n - x, x - y \rangle \quad (31)$$

one deduces by choosing appropriate subsequences

$$\ell(y)^2 = \ell(x)^2 + \|x - y\|^2 \quad (x_{\phi(n)} \rightharpoonup x)$$

and

$$\ell(y)^2 = \ell(x)^2 - \|x - y\|^2 \quad (x_{\psi(n)} \rightharpoonup y)$$

hence  $x = y$ . ■

*Comments*

A Banach space  $X$  satisfies *Opial's condition* if it is reflexive and

$$\limsup_{n \rightarrow \infty} \|x_n - x\| < \limsup_{n \rightarrow \infty} \|x_n - y\| \quad \text{whenever } x_n \rightharpoonup x \neq y. \quad (32)$$

Any uniformly convex Banach space having a weakly continuous duality mapping (in particular, any Hilbert space) satisfies Opial's condition (see [51, Opial]). Opial's Lemma holds in any Banach space satisfying Opial's condition. □

Following [52, Passty], one obtains a more general result:

**Lemma 31** *Let  $\{x_n\}$  be a sequence in  $H$  with step sizes  $\{\lambda_n\}$  and let  $F \subset H$ . Assume (a1) : the sequence  $\|x_n - u\|$  has a limit as  $n \rightarrow \infty$  for each  $u \in F$ . Then the sets  $\Omega[x_n] \cap F$  and  $\Omega[\bar{x}_n] \cap F$  each contains at most one point. In particular if  $\Omega[x_n] \subset F$  (resp.  $\Omega[\bar{x}_n] \subset F$ ), then  $x_n$  (resp.  $\bar{x}_n$ ) converges weakly as  $n \rightarrow \infty$ . A similar result holds for trajectories.*

**Proof.** By (31),  $\langle x_n, x - y \rangle$  converges to some  $m(x, y)$  for any  $x, y \in F$ . If  $u$  and  $v$  belong to  $\Omega[x_n] \cap F$  one obtains  $\langle u, u - v \rangle = \langle v, u - v \rangle$  hence  $u = v$ . Similarly  $\langle \bar{x}_n, x - y \rangle$  converges to  $m(x, y)$ . Thus both  $\Omega[x_n] \cap F$  and  $\Omega[\bar{x}_n] \cap F$  contain at most one point. ■

An alternative proof using (a2) and either (b) or (b') is as follows:

**Lemma 32** *Let  $\{x_n\}$  be a bounded sequence in  $H$  with step sizes  $\{\lambda_n\}$  and let  $F \subset H$  be closed and convex. Assume (a2):  $P_F x_n \rightarrow \zeta$  as  $n \rightarrow \infty$ . Then*

$$\Omega[x_n] \cap F \subset \{\zeta\} \quad \text{and} \quad \Omega[\bar{x}_n] \cap F \subset \{\zeta\}.$$

*In particular, if  $\Omega[x_n] \subset F$  (resp.  $\Omega[\bar{x}_n] \subset F$ ), then  $x_n$  (resp.  $\bar{x}_n$ ) converges weakly to  $\zeta$ . A similar result is true for trajectories.*

**Proof.** By definition of the projection, for each  $u \in F$  one has

$$\langle x_n - P_F x_n, u - P_F x_n \rangle \leq 0.$$

Since  $x_n$  is bounded we deduce that

$$\langle x_n - \zeta, u - \zeta \rangle \leq \rho_n$$

with  $\lim_{n \rightarrow \infty} \rho_n = 0$ . This implies  $\Omega[x_n] \cap F \subset \{\zeta\}$  (if  $v \in \Omega[x_n] \cap F$ , take  $u = v$ ). Similarly

$$\langle \bar{x}_n - \zeta, u - \zeta \rangle \leq \bar{\rho}_n,$$

which gives  $\Omega[\bar{x}_n] \cap F \subset \{\zeta\}$ . ■

A slightly more demanding assumption is the Fejer property:

**(a3)**  $\|u(t) - p\|$  decreases for each  $p \in F$ , or

**(a3')** There exists  $\{\varepsilon_n\} \in \ell^1$  such that  $\|x_{n+1} - u\|^2 \leq \|x_n - u\|^2 + \varepsilon_n$  for all  $u \in F$ .

Then one has the following, from [27, Combettes]:

**Lemma 33** *Any trajectory satisfying (a3) also satisfies (a2). Any sequence satisfying (a3') also satisfies (a2).*

**Proof.** Let  $u(t)$  satisfy **(a3)** and let  $v(t) = P_F u(t)$ . Note first that, using the projection property and **(a3)**

$$\|v(t+h) - u(t+h)\|^2 \leq \|v(t) - u(t+h)\|^2 \leq \|v(t) - u(t)\|^2.$$

hence  $\|v(t) - u(t)\|$  decreases, hence converges.

The parallelogram equality gives

$$\|v(t+h) - v(t)\|^2 + 4 \left\| \frac{v(t+h)+v(t)}{2} - u(t+h) \right\|^2 = 2\|v(t+h) - u(t+h)\|^2 + 2\|v(t) - u(t+h)\|^2.$$

$F$  convex implies  $\left\| \frac{v(t+h)+v(t)}{2} - u(t+h) \right\| \geq \|v(t+h) - u(t+h)\|$ , hence

$$\|v(t+h) - v(t)\|^2 \leq 2 [\|v(t) - u(t)\|^2 - \|v(t+h) - u(t+h)\|^2]$$

so that  $v(t)$  has a strong limit  $v$  as  $t \rightarrow \infty$ .

Now let  $\{x_n\}$  satisfy **(a3')** and write  $y_n = P_F x_n$ . As before, one has

$$\|y_{n+1} - x_{n+1}\|^2 \leq \|y_n - x_{n+1}\|^2 \leq \|y_n - x_n\|^2 + \varepsilon_n$$

so that  $\|y_n - x_n\|^2 + \sum_{m=n}^{+\infty} \varepsilon_m$  is decreasing hence  $\|y_n - x_n\|^2$  converges as well. ■

## 4.2 Characterization of the limit: the asymptotic center

We show here that moreover the weak limit can be characterized.

Given a bounded sequence  $\{x_n\}$  let

$$G(y) = \limsup_{n \rightarrow \infty} \|x_n - y\|^2$$

(for a trajectory  $u(t)$  define  $G(y) = \limsup_{t \rightarrow \infty} \|u(t) - y\|^2$ ). The function  $G(y)$  is continuous, strictly convex and coercive.

Its unique minimizer is called the *asymptotic center* (see [32, Edelstein]) of the sequence (resp. trajectory) and is denoted by  $AC\{x_n\}$  (resp.  $AC\{u(t)\}$ ).

**Proposition 34** Assume **(a1)**. Then  $\Omega[\bar{x}_n] \cap F \subset AC\{x_n\}$ .  
A similar property holds for trajectories.

**Proof.** From (31) one obtains

$$\frac{1}{\sigma_n} \sum_{m=1}^n \lambda_m \|x_m - x\|^2 = \frac{1}{\sigma_n} \sum_{m=1}^n \lambda_m \|x_m - y\|^2 + 2\langle \bar{x}_n - y, y - x \rangle + \|y - x\|^2.$$

Assume  $\bar{x}_{n_k} \rightarrow x \in F$ , then  $\ell(x) = \lim_{n \rightarrow \infty} \|x_n - x\|$  exists by **(a1)**. Therefore,

$$G(x) = \ell(x)^2 \leq \limsup_{k \rightarrow \infty} \left[ \frac{1}{\sigma_{n_k}} \sum_{m=1}^{n_k} \lambda_m \|x_m - y\|^2 \right] - \|x - y\|^2 \leq G(y) - \|x - y\|^2$$

for each  $y \in H$  so that  $x = AC\{x_n\}$ . ■

### 4.3 Characterization of the weak convergence

In this section we use the fact that the trajectories or sequences are generated through a maximal monotone operator and give conditions which are equivalent to weak convergence and do not involve the limit.

Let us consider first the case  $A = I - T$ , where  $T$  is non expansive and defined on  $H$ . The following result is in [55, Pazy]:

**Proposition 35** The sequence  $T^n x$  converges weakly if, and only if,  $\mathcal{S} \neq \emptyset$  and  $\Omega[T^n x] \subset \mathcal{S}$ .

**Proof.** Assume  $\mathcal{S} \neq \emptyset$ . Given  $u \in \mathcal{S}$ , the sequence  $\|T^n x - u\|$  is decreasing and thus convergent. In particular,  $T^n x$  is bounded. By Lemma 31, the fact that  $\Omega[T^n x] \subset \mathcal{S}$  implies that  $T^n x$  converges weakly. Conversely, since the sequence  $\{T^n x\}$  is bounded, the next lemma (or the argument in the proof of Theorem 47) shows that the weak limit points of  $T^n x$  must be in  $\mathcal{S}$ . ■

The following result is interesting in its own right:

**Lemma 36** Assume the sequence  $U_n x = \frac{1}{n}(x + Tx + \dots + T^{n-1}x)$  is bounded. Then  $\emptyset \neq \Omega[U_n x] \subset \mathcal{S}$ .

**Proof.** For any  $y \in H$  one has

$$\begin{aligned} 0 &\leq \|T^k x - y\|^2 - \|T^{k+1} x - Ty\|^2 \\ &= \|T^k x - Ty\|^2 - \|T^{k+1} x - Ty\|^2 + \|Ty - y\|^2 + 2\langle T^k x - Ty, Ty - y \rangle. \end{aligned}$$

By taking the average we obtain

$$0 \leq \frac{1}{n} \|x - Ty\|^2 + \|Ty - y\|^2 + 2\langle U_n x - Ty, Ty - y \rangle.$$

Therefore, if  $p \in \Omega[U_n x]$ , we deduce as  $n \rightarrow \infty$  that

$$0 \leq \|Ty - y\|^2 + 2\langle p - Ty, Ty - y \rangle.$$

In particular, taking  $y = p$  leads to  $\|Tp - p\|^2 \leq 0$  and so  $p \in \mathcal{S}$ . ■

The preceding result shows that  $\Omega[U_n x] \subset \mathcal{S}$  which *a posteriori* implies  $\mathcal{S} \neq \emptyset$ . If one assumes that  $\mathcal{S}$  is nonempty, it is possible to prove that the weak limits of the sequence must belong to  $\mathcal{S}$ :

**Lemma 37** Assume **(a1)**. Then  $T^n x \rightarrow p$  implies  $p \in \mathcal{S}$ .

**Proof.** For any  $y \in H$  and  $u \in F$

$$\begin{aligned} 0 &\leq \|T^k x - y\|^2 - \|T^{k+1} x - Ty\|^2 \\ &= \|T^k x - u\|^2 - \|T^{k+1} x - u\|^2 + \|u - y\|^2 - \|u - Ty\|^2 \\ &\quad + 2\langle T^k x - u, u - y \rangle - 2\langle T^{k+1} x - u, u - Ty \rangle. \end{aligned}$$

Take  $y = p$  and let  $k \rightarrow \infty$ . Since  $\lim_{k \rightarrow \infty} \|T^k x - u\|$  exists we get

$$0 \leq \|u - p\|^2 - \|u - Tp\|^2 + 2\langle p - u, u - p \rangle - 2\langle p - u, u - Tp \rangle,$$

which is precisely  $\|Tp - p\|^2 \leq 0$  and implies  $p \in \mathcal{S}$ . ■

Following [56, Pazy], one obtains the continuous counterpart of Proposition 35:

**Proposition 38** *The trajectory  $S_t x$  converges weakly if, and only if,  $\mathcal{S} \neq \emptyset$  and  $\Omega[S_t x] \subset \mathcal{S}$ .*

**Proof.** Assume  $\mathcal{S} \neq \emptyset$ . By Corollary 7 and Lemma 31,  $\Omega[S_t x] \subset \mathcal{S}$  implies  $S_t x$  converges weakly. It remains to prove that if  $S_t x \rightharpoonup y$  then  $y \in \mathcal{S}$ . Recall that (6) says that for any  $[u, w] \in A$

$$\|S_t x - u\|^2 - \|x - u\|^2 \leq 2 \int_0^t \langle w, u - S_s x \rangle ds.$$

Since  $S_t x$  is bounded, it suffices to divide by  $t$  and let  $t \rightarrow \infty$  to obtain

$$0 \leq \langle w, u - y \rangle$$

so that  $y \in \mathcal{S}$  by maximality. ■

Note that the proof uses the generator  $A$  (compare to the proof of the previous Proposition 35).

A last result, due to [24, Bruck], shows that if  $\mathcal{S} \neq \emptyset$ , then weak convergence is equivalent to weak asymptotic regularity. We follow [57, Pazy].

**Proposition 39** *Assume  $\mathcal{S} \neq \emptyset$ . The trajectory  $S_t x$  converges weakly if, and only if,*

$$S_{t+h} x - S_t x \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

*for each  $h \geq 0$ . A similar result holds for the sequence  $T^n x$ .*

**Proof.** For  $u \in \mathcal{S}$  and  $t > s$  we have

$$2\langle S_{s+h} x - u, S_s x - u \rangle - 2\langle S_{t+h} x - u, S_t x - u \rangle \leq \|S_{s+h} x - u\|^2 - \|S_{t+h} x - u\|^2 + \|u - S_s x\|^2 - \|u - S_t x\|^2.$$

Let  $w \in \Omega[S_t x]$  and  $h_k \rightarrow \infty$  with  $S_{t+h_k} x \rightharpoonup w$ . Then  $S_{s+h_k} x \rightharpoonup w$  as well by weak asymptotic regularity. Thus we obtain

$$2\langle w - u, S_s x - S_t x \rangle \leq \|u - S_s x\|^2 - \|u - S_t x\|^2.$$

By **(a1)**,  $\langle w - u, S_t x \rangle$  has a limit  $L(w)$ . In particular  $w' \in \Omega[S_t x]$  implies  $\langle w - u, w' \rangle = L(w)$  so that  $\langle w - u, w' - w \rangle = 0$ . Hence by symmetry  $\langle w' - u, w - w' \rangle = 0$ , thus  $w = w'$  and  $\Omega[S_t x]$  is reduced to one point. ■

## 5 Weak convergence in average

We now turn to the study of the asymptotic behavior of the averages.

## 5.1 Continuous dynamics

Consider  $x \in \overline{D(A)}$ . In order to use the semigroup notation, let us introduce

$$\sigma_t x = \frac{1}{t} \int_0^t S_s x \, ds.^2$$

In order to prove that  $\sigma_t x$  converges weakly as  $t \rightarrow \infty$  we follow the ideas in [12, Baillon and Brézis]. We first prove that the projection  $P_S S_t x$  converges strongly to some  $v$  (**a2**), next that weak cluster points of  $\sigma_t x$  are in  $\mathcal{S}$  (**b'**), and finally use Lemma 32 to conclude that  $\sigma_t x$  converges weakly to  $v$ .

**Lemma 40** *Assume  $\mathcal{S} \neq \emptyset$ . Then  $P_S S_t x$  converges strongly.*

**Proof.** By Lemma 6,  $S_t x$  satisfies (**a3**) for  $\mathcal{S}$  and we can use Lemma 33. ■

**Lemma 41**  $\Omega[\sigma_t x] \subset \mathcal{S}$ .

**Proof.** Assume  $\sigma_{t_k} x \rightarrow u$  as  $k \rightarrow \infty$ . Inequality (6) gives

$$2 \int_0^{t_k} \langle w, v - S_t x \rangle \, dt \geq \|S_{t_k} x - v\|^2 - \|x - v\|^2 \geq -\|x - v\|^2$$

for each  $[v, w] \in A$ . Divide by  $t_k$  and let  $k \rightarrow \infty$ . We get  $\langle w, v - u \rangle \geq 0$  so that  $0 \in Au$  by maximality. ■

*Comments*

Lemma 41 implies that if  $\mathcal{S} = \emptyset$  then  $\|\sigma_t x\| \rightarrow \infty$  for every  $x \in \overline{D(A)}$  as  $t \rightarrow \infty$ . On the other hand, if  $\mathcal{S} \neq \emptyset$  then every trajectory  $S_t x$  is bounded, so  $\sigma_t x$  is bounded for all  $x \in \overline{D(A)}$ . □

Using Lemma 32, Lemma 40 and Lemma 41 we finally obtain

**Theorem 42** *If  $\mathcal{S} \neq \emptyset$ , then  $\sigma_t x$  converges weakly to  $v = \lim_{t \rightarrow \infty} P_S S_t x$ .*

As a consequence of Proposition 34 and Lemma 41 one has

**Proposition 43** *If  $\mathcal{S} \neq \emptyset$ , the weak limit  $w\text{-}\lim_{t \rightarrow \infty} \sigma_t x$  is the asymptotic center  $AC\{S_t x\}$ .*

*Comments*

Weak convergence in average is still true in uniformly convex Banach space with Fréchet-differentiable norm (see [60, Reich]) or satisfying Opial's condition (see [35, Hirano]). □

If  $A = \partial f$  with  $f \in \Gamma_0(H)$ , convergence in average guarantees the convergence of the trajectory (see [22, Bruck]):

**Proposition 44** *If  $A = \partial f$  then  $\lim_{t \rightarrow \infty} \left\| u(t) - \frac{1}{t} \int_0^t u(s) \, ds \right\| = 0$ .*

**Proof.** Integration by parts gives  $u(t) - \frac{1}{t} \int_0^t u(s) \, ds = \frac{1}{t} \int_0^t s \dot{u}(s) \, ds$ .  $\|\dot{u}(t)\|$  being decreasing by Proposition 9, one has

$$\int_{t/2}^t s \|\dot{u}(s)\|^2 \, ds \geq \|\dot{u}(t)\|^2 \int_{t/2}^t s \, ds = \frac{3}{8} t^2 \|\dot{u}(t)\|^2.$$

But in the case  $A = \partial f$ , the function  $t \mapsto t \|\dot{u}(t)\|^2$  is in  $L^1(0, \infty)$  (see [18, Brézis]) which implies  $\lim_{t \rightarrow \infty} t \|\dot{u}(t)\| = 0$  and the result follows. ■

It is known that both the trajectory and the average converge weakly (Theorems 42 and 52). The preceding result implies, in particular, that the average cannot converge strongly unless the trajectory itself does.

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<sup>2</sup>More generally  $\sigma_n x = \int_0^\infty S_s x a_n(s) \, ds$  where  $a_n$  is the density of a positive probability measure on  $\mathbf{R}^+$ , which is assumed to be of bounded variation with  $\int_0^\infty |da_n| \rightarrow 0$ . For example  $a_n(s) = \frac{1}{n} \chi_{[0, n]}(s)$ .

## 5.2 Proximal sequences

Consider a proximal sequence  $\{x_n\}$  in  $H$  along with step sizes  $\{\lambda_n\}$ , and recall that  $\bar{x}_n = \frac{1}{\sigma_n} \sum_{k=1}^n \lambda_k x_k$ . The next result was presented in [43, Lions]:

**Theorem 45** *Let  $S \neq \emptyset$ . Then  $\{x_n\}$  converges weakly in average to a point in  $S$ .*

**Proof.** Let us check the conditions of Lemma 31 with  $F = S$ : **(a3)** follows from Corollary 16, while **(b')** follows from Lemma 46 below. ■

The following is a discrete-time counterpart of Lemma 41:

**Lemma 46**  $\Omega[\bar{x}_n] \subset S$ .

**Proof.** Take  $[u, v] \in A$  and use inequality (11) to get

$$\|x_0 - u\|^2 \geq 2 \sum_{n=1}^N \lambda_n \langle v, x_n - u \rangle.$$

If  $\bar{x}_{n_k} \rightharpoonup \bar{x}$ , then dividing by  $\sigma_{n_k}$  we obtain  $\langle v, u - \bar{x} \rangle \geq 0$ , whence  $\bar{x} \in S$  by maximality. ■

As for the continuous trajectory, the weak limit of the sequence  $\{\bar{x}_n\}$  is  $AC\{x_n\}$ .

*Comments*

If  $\{\lambda_n\} \notin \ell^2$  one proves a stronger result: the sequence  $\{x_n\}$  converges weakly (Theorem 53). □

The extension to the sum of two operators is in [52, Passty].

## 5.3 Euler sequences

For nonexpansive mappings, weak convergence in average of the discrete iterates was established in [7, Baillon]. The proof is again of the form **(a3)** and **(b')** but note that the property  $S \neq \emptyset$  is not assumed but obtained during the proof.

**Theorem 47** *Let  $T$  be a nonexpansive mapping on a bounded closed convex subset  $C$  of  $H$ . For every  $z \in C$  the sequence  $z_n = T^n z$  converges weakly in average to a fixed point of  $T$ , which is the strong limit of the sequence  $P_S T^n z$ .*

**Proof.** Note that for any  $a$  and  $a^i, i = 0, \dots, n-1$ , in  $H$ , the quantity

$$\left\| a - \frac{1}{n} \sum_{i=0}^{n-1} a^i \right\|^2 - \frac{1}{n} \sum_{i=0}^{n-1} \|a - a^i\|^2$$

is independent of  $a$ . Hence with  $U_n z = \frac{1}{n}(z + Tz + \dots + T^{n-1}z)$  one has

$$\begin{aligned} \|TU_n z - U_n z\|^2 &= \frac{1}{n} \sum_{i=0}^{n-1} \|TU_n z - T^i z\|^2 - \frac{1}{n} \sum_{i=0}^{n-1} \|U_n z - T^i z\|^2 \\ &\leq \frac{1}{n} (\|TU_n z - z\|^2 - \|U_n z - T^{n-1} z\|^2) \end{aligned}$$

so that

$$\|TU_n z - U_n z\| \leq \frac{1}{\sqrt{n}} \|TU_n z - z\|.$$

Thus  $TU_n z - U_n z \rightarrow 0$  and if  $U_n z \rightarrow u$  then  $Tu = u$  by Remark 5. It follows that  $\Omega[U_n z] \subset \mathcal{S}$ , which is **(b')** and  $\mathcal{S} \neq \emptyset$ .

Since  $\|T^m z - u\|$  decreases for  $u \in \mathcal{S}$ , **(a3)** holds. By Lemma 33 the strong limit  $V$  of  $P_S$  exists and **(b')** implies that  $\Omega[U_n z] = \{V\}$  by Lemma 32.  $\blacksquare$

#### Comments

The conclusion of Theorem 47 holds also if  $X$  is uniformly convex with Fréchet-differentiable norm and  $\lambda_n \rightarrow 1$  or if  $X$  is superreflexive ([60, Reich]).  $\square$

Following an idea of Konishi (see [11, Baillon]) one can prove that the ergodic theorem for nonexpansive mappings implies in fact the analogous results for the semi-group:

**Proposition 48** *Theorem 47 implies Theorem 42.*

**Proof.** Let  $0 < h < t$  and  $n = [t/h]$  the integer part of  $t/h$  and set  $T_h = S_h$  and  $U_n^h x = \frac{1}{n} \sum_{m=0}^{n-1} T_h^m x$ . One deduces that

$$\|\sigma_t x - n h U_n^h x\| \leq n \int_0^h \|S_s x - x\| ds + M h,$$

where  $\|S_s x\| \leq M$ . Thus

$$\|\sigma_t x - U_n^h x\| \leq \frac{1}{h} \int_0^h \|S_s x - x\| ds + \frac{2M}{n}.$$

But as  $t \rightarrow +\infty$ ,  $U_n^h x$  converges weakly to a fixed point  $u_h$  of  $T_h$  by Theorem 47.

Let us now prove that  $u_h$  is a Cauchy net as  $h \rightarrow 0$ . Given  $0 < h, h' < t$ ,  $n = [t/h]$ ,  $n' = [t/h']$  one has

$$\|U_n^h x - U_{n'}^{h'} x\| \leq \frac{1}{h} \int_0^h \|S_s x - x\| ds + \frac{2M}{n} + \frac{1}{h'} \int_0^{h'} \|S_s x - x\| ds + \frac{2M}{n'}.$$

By the weak lower-semicontinuity of the norm, as  $t \rightarrow +\infty$  we have

$$\|u_h - u_{h'}\| \leq \frac{1}{h} \int_0^h \|S_s x - x\| ds + \frac{1}{h'} \int_0^{h'} \|S_s x - x\| ds.$$

Since  $\|S_s x - x\| \rightarrow 0$  as  $s \rightarrow 0$ ,  $u_h$  is a Cauchy net, that converges to some  $u$ . But  $S_{mh} u_h = u_h$ , so that given  $s$  and  $h = s/m$ , one has  $S_s u_h = u_h$ . As  $m \rightarrow +\infty$  this implies  $S_s u = u$ , thus  $u \in \mathcal{S}$ . Now write, given  $y \in H$

$$|\langle \sigma_t x - u, y \rangle| \leq |\langle \sigma_t x - U_n^h x, y \rangle| + |\langle U_n^h x - u_h, y \rangle| + \|u_h - u\| \|y\|$$

hence

$$|\langle \sigma_t x - u, y \rangle| \leq \left( \frac{1}{h} \int_0^h \|S_s x - x\| ds + \frac{2M}{n} \right) \|y\| + |\langle U_n^h x - u_h, y \rangle| + \|u_h - u\| \|y\|.$$

It follows that

$$\limsup_{t \rightarrow +\infty} |\langle \sigma_t x - u, y \rangle| \leq \left( \frac{1}{h} \int_0^h \|S_s x - x\| ds \right) \|y\| + \|u_h - u\| \|y\|$$

for all  $h > 0$ . Letting  $h \rightarrow 0$  we obtain  $\sigma_t x \rightarrow u$ .  $\blacksquare$

Recall that  $\bar{z}_n = \frac{1}{\sigma_n} \sum_{k=1}^n \lambda_k z_k$ , where  $z_n$  is given in (17). A general result on convergence in average is the following from [23, Bruck]:

**Theorem 49** *Assume  $\sum \|z_n - z_{n-1}\|^2 < \infty$ . If  $\mathcal{S} \neq \emptyset$ , then  $z_n$  converges weakly in average to  $\zeta = \lim_{n \rightarrow \infty} P_S z_n$ . Otherwise  $\lim_{n \rightarrow \infty} \|\bar{z}_n\| = \infty$ .*



**Proof.** We first prove that  $\Omega[\bar{z}_n] \subset \mathcal{S}$ , which is **(b')**. Then we show, if  $\mathcal{S}$  is non empty, that the projections  $\zeta_n = P_{\mathcal{S}}z_n$  converge strongly to some  $\zeta \in \mathcal{S}$ , which is **(a2)**. Finally we verify that  $\zeta$  is the only weak cluster point of the bounded sequence  $\{\bar{z}_n\}$ .

First, let  $[u, v] \in A$  and use (20) to deduce that

$$\|z_{n+1} - u\|^2 \leq \|z_n - u\|^2 + 2\lambda_n \langle v, u - z_n \rangle + \lambda_n^2 \|w_n\|^2.$$

Summing up, neglecting the positive term of the telescopic sum on the left-hand side and dividing by  $\sigma_n$  we get

$$0 \leq \frac{\|z_1 - u\|^2}{\sigma_n} + \frac{1}{\sigma_n} \sum_{k=1}^n \|z_{k+1} - z_k\|^2 + 2\langle v, u - \bar{z}_n \rangle.$$

Therefore  $\liminf_{n \rightarrow \infty} \langle v, u - \bar{z}_n \rangle \geq 0$  and every weak cluster point of  $\{\bar{z}_n\}$  lies in  $\mathcal{S}$ , by maximality. Notice that this is **(b')**, hence the counterpart of Lemma 41 and Lemma 46.

Next, take  $u \in \mathcal{S}$ . From equation (20) we get

$$\|z_{n+1} - u\|^2 \leq \|z_n - u\|^2 + \|\lambda_n w_n\|^2. \quad (33)$$

This implies the convergence of  $\|z_{n+1} - u\|^2$  hence **(a1)** which ends the proof of convergence in  $\mathcal{S}$  by using Lemma 31.

Note that the sequence  $z_n$  satisfies **(a3')**, hence **(a2)** by Lemma 33. The last result now follows from Lemma 32. ■

For a similar proof with two operators and forward-backward procedure see [52, Passty].

The following result due to [57, Pazy] of **(b')** leads to a unified proof of weak convergence in average for contractions in the discrete (Theorem 47) or continuous case (Theorem 42). Note that the first step assumes  $\mathcal{S} \neq \emptyset$ , then one uses **(a1)** to achieve the result but the generator  $A$  is not used.

**Proposition 50** *Assume  $\mathcal{S} \neq \emptyset$ , then  $\Omega[\sigma_t x] \subset \mathcal{S}$ .*

**Proof.** For  $t, h \geq 0$  we have, for any  $y \in H$

$$\begin{aligned} 0 &\leq \|S_t x - y\|^2 - \|S_{t+h} x - S_h y\|^2 \\ &= \|S_t x - S_h y\|^2 - \|S_{t+h} x - S_h y\|^2 + 2\langle S_t x - S_h y, S_h y - y \rangle + \|S_h y - y\|^2. \end{aligned}$$

By taking the average we deduce that

$$0 \leq \frac{1}{t} \int_0^t [\|S_s x - S_h y\|^2 - \|S_{s+h} x - S_h y\|^2] ds + 2\langle \sigma_t x - S_h y, S_h y - y \rangle + \|S_h y - y\|^2.$$

Since  $\mathcal{S} \neq \emptyset$ ,  $\|S_t x - S_h y\|$  is bounded. Letting  $t \rightarrow +\infty$ , it follows that for any  $p \in \Omega[\sigma_t x]$ , any  $h \geq 0$  and any  $y \in H$

$$0 \leq 2\langle p - S_h y, S_h y - y \rangle + \|S_h y - y\|^2.$$

Finally take  $y = p$  so that  $p = S_h p$ , which means  $p \in \mathcal{S}$ . ■

*Comments*

The only use of  $\mathcal{S} \neq \emptyset$  is to guarantee the boundedness of  $S_h y$  (compare with Lemma 36). □

## 6 Weak convergence

Not all maximal monotone operators generate weakly convergent trajectories.

**Example 4** Let  $R : \mathbf{R}^2 \rightarrow \mathbf{R}^2$  be the clockwise  $\pi/2$ -rotation and consider the evolution scheme defined by the differential equation:

$$\dot{u}(t) + R(u(t)) = 0.$$

Note that  $\mathcal{S} = \{0\}$ . The orbit starting at time  $t = 0$  from the point  $x = r_0(\cos(\theta_0), \sin(\theta_0))$  with  $r_0 > 0$  is described by  $u(t) = r_0(\cos(t + \theta_0), \sin(t + \theta_0))$ , which is bounded but does not have a limit as  $t \rightarrow \infty$ . However, the average  $\frac{1}{t} \int_0^t u(s) ds$  converges to 0 as  $t \rightarrow \infty$ , by Theorem 42.

Now let  $x_n = r_n(\cos \theta_n, \sin \theta_n)$  satisfy

$$\frac{x_{n+1} - x_n}{\lambda_n} = R(x_{n+1}).$$

We have  $r_{n+1}^2 = \prod_{k=1}^n (1 + \lambda_k^2)^{-1} r_0^2$  and  $\theta_n = \theta_0 + \sum_{k=1}^n \arctan(\lambda_k)$ . The sequence  $r_n$  is decreasing. If  $\lambda_n \notin \ell^2$  then  $\lim_{n \rightarrow \infty} x_n = 0$ ; otherwise it stays bounded away from zero. On the other hand, the argument  $\theta_n$  is increasing. It converges if  $\lambda_n \in \ell^1$  and diverges otherwise. Observe also that  $x_n$  converges in average to 0 as  $n \rightarrow \infty$ , by Theorem 45.

Finally, let  $z_n = \rho_n(\cos \phi_n, \sin \phi_n)$  satisfy

$$\frac{z_{n+1} - z_n}{\lambda_n} = R(z_n).$$

Here  $\rho_{n+1}^2 = \prod_{k=1}^n (1 + \lambda_k^2) \rho_0^2$  and  $\phi_n = \phi_0 + \sum_{k=1}^n \arctan(\lambda_k)$ . In this case the sequence  $\rho_n$  is increasing. It remains bounded and is convergent if, and only if,  $\lambda_n \in \ell^2$ . The argument  $\phi_n$  is increasing as well. It converges if  $\lambda_n \in \ell^1$  and diverges otherwise. As before,  $z_n$  converges in average to 0 as  $n \rightarrow \infty$  by Theorem 49. To see this simply observe that  $w_n = \frac{z_{n+1} - z_n}{\lambda_n} = R(z_n)$  is bounded and  $\lambda_n \in \ell^2$ , so that  $\sum \|z_{n+1} - z_n\|^2 < \infty$ .  $\square$

## Tools

Assuming  $\mathcal{S}$  non empty and using Lemma 31, by virtue of Corollary 7 (resp. Corollary 16 and Corollary 23), in order to prove weak convergence of  $u(t)$  (resp.  $x_n, z_n$ ), it suffices to verify that its set of weak cluster points lie in  $\mathcal{S}$  (condition (b)). The key tool is the concept of *demipositivity*, first developed in [22, Bruck].

A maximal monotone operator  $A$  is *demipositive* if there exists  $w \in \mathcal{S}$  such that for every sequence  $\{u_n\} \in D(A)$  converging weakly to  $u$  and every bounded sequence  $\{v_n\}$  such that  $v_n \in Au_n$

$$\langle v_n, u_n - w \rangle \rightarrow 0 \quad \text{implies} \quad u \in \mathcal{S}. \quad (34)$$

The following gathers examples from [22, Bruck] and [54, Pazy] (not the last two, which are trivial):

**Proposition 51** *Each of the following conditions is sufficient for a maximal monotone operator  $A$  to be demipositive:*

1.  $A = \partial\phi$ , where  $\phi$  is a proper lower-semicontinuous convex function having minimizers ( $\mathcal{S} \neq \emptyset$ ).
2.  $A = I - T$ , where  $T$  is nonexpansive and has a fixed point ( $\mathcal{S} \neq \emptyset$ ).
3. The set  $\mathcal{S}$  has nonempty interior.
4.  $A$  is odd and firmly positive, which means that there is  $w \in \mathcal{S}$  such that  $v \in Au$  and  $\langle v, u - w \rangle = 0$  together imply  $0 \in Au$ .
5.  $A$  is firmly positive and sequentially weakly closed (its graph is sequentially weak/weak closed).
6.  $\mathcal{S} \neq \emptyset$  and  $A$  is 3-monotone, which means that  $\sum_{n=1}^3 \langle y_n, x_n - x_{n-1} \rangle \geq 0$  for every set  $\{[x_n, y_n] \mid 1 \leq n \leq 3\} \subset A$  ( $x_0 \equiv x_N$ ).
7.  $A$  is strongly monotone:  $\langle x^* - y^*, x - y \rangle \geq \alpha \|x - y\|^2$  for all  $[x, x^*], [y, y^*] \in A$  and some  $\alpha > 0$ .
8.  $\mathcal{S} \neq \emptyset$  and  $A$  is cocoercive:  $\langle x^* - y^*, x - y \rangle \geq \mu \|x^* - y^*\|^2$  for all  $[x, x^*], [y, y^*] \in A$  and some  $\mu > 0$ .

For demipositivity in Banach spaces see [26, Bruck and Reich].

*Comments*

We just mention another assumption that guarantees that the weak cluster points will lie in  $\mathcal{S}$ . Let  $S$  the semi-group generated by  $A$ .  $A$  satisfies condition (L) if

$$\lim_{t \rightarrow \infty} \|A^0 S_t x\| \leq \lim_{t \rightarrow \infty} \left( \frac{1}{h} \|S_{t+h} x - S_t x\| \right)$$

for every  $h > 0$  and  $x \in D(A)$ . The interested reader may find this definition and related results in [54, Pazy]. An equivalent formulation is the following: Denote by  $a^0$  the element of minimal norm in  $\overline{R(A)}$ . Then  $A$  satisfies condition (L) if, and only if, for every  $x \in D(A)$  one has

$$\lim_{t \rightarrow \infty} A^0 S_t x = a^0.$$

Unlike demipositivity, this does not impose *a priori* that  $\mathcal{S} \neq \emptyset$ . For instance, if  $A = \partial f$  with  $f \in \Gamma_0(H)$  or if  $A = I - T$  with  $T$  nonexpansive, then  $A$  satisfies condition (L) but is not demipositive unless  $\mathcal{S} \neq \emptyset$ .

Condition (L) is essentially used in [54, Pazy] to prove that the weak cluster points of the trajectory  $S_t x$  lie in  $\mathcal{S}$ . If  $\mathcal{S} = \emptyset$  one immediately deduces that  $\lim_{t \rightarrow \infty} \|S_t x\| = \infty$ .  $\square$

## 6.1 Continuous dynamics

The following classical result of weak convergence for demipositive operators was proved in [22, Bruck].

**Theorem 52** *If  $A$  is demipositive then  $u(t)$  converges weakly as  $t \rightarrow \infty$  to an element of  $\mathcal{S}$ .*

**Proof.** By Corollary 7 and Opial's Lemma it suffices to prove  $\Omega[u(t)] \subset \mathcal{S}$ , which is (b). Let  $w \in \mathcal{S}$  satisfy (34) and let  $u(t_n) \rightarrow w$  as  $n \rightarrow \infty$ .

Set  $\theta(t) = \frac{1}{2} \|u(t) - w\|^2$ , so that  $\dot{\theta}(t) = \langle \dot{u}(t), u(t) - w \rangle$ . Notice that  $\theta$  is bounded by Corollary 7. Whence  $\dot{\theta} \in L^1$  and  $\dot{u}$  is bounded by Theorem 12. By considering the intervals  $[t_n - 1/n, t_n + 1/n]$  one deduces that there is a sequence  $s_{n_k}$  such that  $\dot{\theta}(s_{n_k}) \rightarrow 0$  as  $k \rightarrow \infty$  and  $u(s_{n_k}) \rightarrow w$ . So  $w \in \mathcal{S}$  by demipositivity.  $\blacksquare$

*Comments*

Theorem 52 was extended in [53, Passty] to the class of  $\varphi$ -demipositive operators.  $\square$

## 6.2 Proximal sequences

A first detailed study of the asymptotic behavior of the proximal sequence  $\{x_n\}$  was performed in [62, Rockafellar], when the step sizes are bounded away from zero. The author also considers an inexact version of the algorithm. The next convergence results under more general hypotheses are investigated in [20, Brézis and Lions].

Recall that  $\sigma_n = \sum_{m \leq n} \lambda_m$  and  $\tau_n = \sum_{m \leq n} \lambda_m^2$ .

**Theorem 53** *Assume  $\mathcal{S} \neq \emptyset$ . If  $\{\lambda_n\} \notin \ell^2$  then  $x_n$  converges weakly to some  $x^* \in \mathcal{S}$ . Moreover,  $\|y_n\| \leq d(x_0, \mathcal{S}) \tau_n^{-1/2}$ .*

**Proof.** By Lemmas 13 and 15, we have for any  $x \in \mathcal{S}$

$$\|y_n\|^2 \tau_n \leq \sum_{k \leq n} \lambda_k^2 \|y_k\|^2 \leq \|x_0 - x\|^2.$$

$\tau_n \rightarrow \infty$  implies  $\|y_n\| \rightarrow 0$ . Since  $-y_n \in Ax_n$ , we deduce that  $\Omega[x_n] \subset \mathcal{S}$ , which is (b), by Proposition 4. We conclude by Corollary 16 and Opial's Lemma 30.  $\blacksquare$

The following result, adding the demipositivity hypothesis, is also from [20, Brézis and Lions]:

**Theorem 54** *If  $A$  is demipositive then  $x_n$  converges weakly to some  $x^* \in \mathcal{S}$ .*

**Proof.** As above, using Corollary 16 the result follows from Opial's Lemma 30 if  $\Omega[x_n] \subset \mathcal{S}$ , which is (b). Let  $x_{n_k} \rightharpoonup x$  and  $w$  be the element in  $\mathcal{S}$  used in the definition of demipositivity (34). Using Lemma 55 below we construct another subsequence  $\{x_{m_k}\}$  such that both  $\|x_{m_k} - x_{n_k}\|$  and  $\langle x_{m_k} - w, y_{m_k} \rangle$  tend to 0 as  $k \rightarrow \infty$ . Since  $x_{m_k} \rightharpoonup x$  and  $A$  is demipositive,  $x$  must belong to  $\mathcal{S}$ . ■

**Lemma 55** *Let  $\{x_n\}$  be a proximal sequence and  $w \in \mathcal{S}$ . For each  $\varepsilon > 0$ , there is  $N$  such that: for any  $n \geq N$ , there exists  $m \in \mathbf{N}$  satisfying  $N \leq m \leq n$ ,  $\|x_m - x_n\| \leq \varepsilon$  and  $\langle -y_m, x_m - w \rangle \leq \varepsilon$ .*

**Proof.** For each  $w \in \mathcal{S}$ , (12) implies that  $\|x_{k-1} - w\|^2 \geq \|x_k - w\|^2 + 2\lambda_k \langle -y_k, x_k - w \rangle$  and so

$$\sum_k \lambda_k \langle y_k, w - x_k \rangle < \infty \quad (35)$$

where all terms are nonnegative by monotonicity. Given  $\varepsilon > 0$ , define  $P = \{k \in \mathbf{N} \mid \langle y_k, w - x_k \rangle \geq \varepsilon\}$  so that  $\sum_{k \in P} \lambda_k < \infty$ . Since  $\|x_{k-1} - x_k\| = \lambda_k \|y_k\|$ , Lemma 13 implies  $\sum_{k \in P} \|x_{k-1} - x_k\| < \infty$ . Let  $N_1$  so that  $\sum_{k \in P, k \geq N_1} \|x_{k-1} - x_k\| < \varepsilon$ . By virtue of (35), since  $\{\lambda_n\} \notin \ell^1$  there is  $N \geq N_1$  with  $\langle y_N, w - x_N \rangle \leq \varepsilon$ . Consider  $n \geq N$ : if  $n \notin P$  we choose  $m = n$ . If  $n \in P$ , let  $m = \max\{k < n \mid k \notin P\}$ . Since  $m \geq N_1$  and all integers between  $m$  and  $n$  are in  $P$ , we have  $\|x_m - x_n\| \leq \sum_{m < k \leq n} \|x_{k-1} - x_k\| \leq \varepsilon$ . ■

*Comments*

1. If  $A = \partial f$  then the result follows from Corollary 16 and Proposition 26.
2. Theorem 54 is still true if the sequence satisfies  $\|x_n - (I + \lambda_n A)^{-1} x_{n-1}\| \leq \varepsilon_n$  with  $\sum \varepsilon_n < \infty$ . This is proved in [20, Brézis and Lions] and can also be derived using asymptotic equivalence results in Section 8 (see [2, 3]).
3. In uniformly convex Banach spaces with Fréchet differentiable norm there is weak convergence in the following cases (see [61, Reich]):
  - (a)  $\{\lambda_n\}$  does not converge to zero, or
  - (b) The modulus of convexity of the space satisfies  $\delta(\varepsilon) \geq K\varepsilon^p$  for some  $K > 0$  and  $p \geq 2$  and  $\sum \lambda_n^p = \infty$ .
4. Demipositive can be replaced by  $\varphi$ -demipositive (see [53, Passty]). □

### 6.3 Euler sequences

As in Theorems 52 and 54, for Euler sequences we have the following:

**Theorem 56** *Let  $A$  be demipositive and assume  $\{\lambda_n\} \in \ell^2$  and  $\{w_n\}$  bounded. Then  $z_n$  converges weakly to some  $z \in \mathcal{S}$ .*

**Proof.** If  $y \in \mathcal{S}$ , Corollary 23 shows that the sequence  $\|z_n - y\|$  is convergent. On the other hand, equality (21) and the hypothesis implies  $\sum_{n \geq 1} \lambda_n \langle w_n, y - z_n \rangle < \infty$  which plays the role of (35). One concludes as in Theorem 54 proving an analogue of Lemma 55 (and using the fact that  $w_n$  is bounded). ■

*Comments*

The previous result from [26, Bruck and Reich] works for demipositive operators in “a few” Banach spaces, namely  $X = L^{2m}$ ,  $m \in \mathbf{N}$  or  $X = \ell^p$ ,  $p \in (1, \infty)$ . □

A related result from [61, Reich] is the following (and holds in uniformly convex Banach spaces with Fréchet-differentiable norm):

**Proposition 57** Let  $T$  be nonexpansive,  $A = I - T$  and  $\{\lambda_n\}$  satisfying  $0 \leq \lambda_n \leq 1$  and  $\sum \lambda_n(1 - \lambda_n) = \infty$ . If  $S \neq \emptyset$  then  $\{z_n\}$  converges weakly to a point in  $S$ .

If  $A = \partial f$  with  $f \in \Gamma_0(H)$  and  $\dim(H) < \infty$  one can circumvent the difficulties of Lemma 55 and provide a simpler proof of Theorem 56.

**Theorem 58** Assume  $A = \partial f$  with  $f \in \Gamma_0(H)$ ,  $S \neq \emptyset$  and  $\dim(H) < \infty$ . If  $\sum \|z_n - z_{n-1}\|^2 < \infty$  then  $z_n$  converges to a minimizer of  $f$ .

**Proof.** Lemma 28 gives  $\liminf_{n \rightarrow \infty} f(z_n) = f^*$ . Since  $\{z_n\}$  is bounded and the space is finite dimensional, there is a subsequence  $\{z_{n_k}\}$  such that  $\lim_{k \rightarrow \infty} f(z_{n_k}) = f^*$  and  $\lim_{k \rightarrow \infty} \|z_{n_k} - z\| = 0$  for some  $z \in H$ . Since  $z$  must be in  $S$  by lower-semicontinuity, Corollary 23 implies  $\lim_{n \rightarrow \infty} \|z_n - z\| = 0$ , which means  $z_n$  converges to  $z$ . ■

The preceding result from [63, Shepilov] was pointed out to the authors by R. Cominetti.

## 7 Strong convergence

Even if  $A = \partial f$  with  $f \in \Gamma_0(H)$  having minimizers, the trajectory  $u(t)$  need not converge strongly as  $t \rightarrow \infty$ . This is shown by Baillon's example in [10, Baillon]: the author defines a function  $f \in \Gamma_0(\ell^2)$  having minimizers and proves that the trajectories converge weakly but not strongly.

This is also true for the proximal point algorithm. Even if  $A = \partial f$  with  $f \in \Gamma_0(H)$  having minimizers, a sequence satisfying (8) need not converge strongly. This was proved in [33, Güler] using Baillon's example and the equivalence techniques from [53, Passty]. A different example of this type can be found in [14, Bauschke et al] and can be retranslated to provide a new counterexample for strong convergence of the continuous trajectory, different from that of Baillon.

### Conditions

We introduce here a series of conditions, mainly of geometric nature, that will be used to obtain strong convergence of the process in the continuous or discrete set-up.

**Strong monotonicity.** Let  $\alpha > 0$ . An operator  $A$  is  $\alpha$ -strongly monotone if for all  $[x, x^*], [y, y^*] \in A$  one has

$$\langle x^* - y^*, x - y \rangle \geq \alpha \|x - y\|^2.$$

Observe that if  $A$  is strongly monotone and  $Ax \cap Ay \neq \emptyset$ , then  $x = y$ . If  $A$  is  $\alpha$ -strongly monotone then  $J_{1/\alpha}^A$  is a strict contraction. Therefore it has a fixed point  $p$  and only one, say  $S = \{p\}$ . Strongly monotone operators are demipositive.

Clearly, if  $A$  is monotone, then  $A + \alpha I$  is  $\alpha$ -strongly monotone. Also, subdifferentials of proper, lower-semicontinuous strongly convex functions are strongly monotone.

A weaker notion of strong monotonicity, found for instance in [54, Pazy], is the following:  $A$  is  $\alpha$ -strongly monotone if  $S \neq \emptyset$  and

$$\langle A^0 x, x - P_S x \rangle \geq \alpha \|x - P_S x\|^2$$

for every  $x \in D(A)$ . In this case the set  $S$  need not be a singleton. Proposition 59 below also holds if  $A$  is strongly monotone in this sense but the proof is more involved.

**Solution set  $S$  with nonempty interior.** If  $p \in \text{int}S$  then there is  $r > 0$  such that the ball  $B(p, r)$  of radius  $r$  centered at  $p$  is contained in  $S$ . Then  $\langle u^*, u - p + rh \rangle \geq 0$  for all  $[u, u^*] \in A$  and all  $h \in H$  with  $\|h\| \leq 1$ . Therefore  $\langle u^*, u - p \rangle \geq r \langle u^*, -h \rangle$  and

$$r \|u^*\| = r \sup_{\|h\| \leq 1} \langle u^*, -h \rangle \leq \langle u^*, u - p \rangle. \quad (36)$$

**The NR convergence condition.** A maximal monotone operator  $A$  on  $H$  satisfies the *NR convergence condition* if  $S \neq \emptyset$  and for every bounded sequence  $[x_n, y_n] \in A$  one has

$$\liminf_{n \rightarrow \infty} \langle y_n, x_n - P_S x_n \rangle = 0 \quad \text{implies} \quad \liminf_{n \rightarrow \infty} \|x_n - P_S x_n\| = 0.$$

Strongly monotone operators satisfy this condition. So do operators having compact resolvent (see below) and those satisfying  $\langle y, x - P_S x \rangle > 0$  for all  $[x, y] \in A$  such that  $x \notin \mathcal{S}$ .

The NR convergence condition can be easily stated in a Banach space  $X$  by means of the duality mapping. The results below hold when both  $X$  and  $X^*$  are uniformly convex. The interested reader can consult [50, Nevanlinna and Reich] and [26, Bruck and Reich].

**Strong precompactness.** The strong  $\omega$ -limit set of a trajectory  $u : [0, \infty) \rightarrow H$  is the set  $\omega[u(t)] = \bigcap_{t>0} \overline{\{u(s) : s \geq t\}}$ . For a sequence  $\{x_n\}$  it is defined by  $\omega[x_n] = \bigcap_{n \in \mathbf{N}} \overline{\{x_k : k \geq n\}}$ .

In the setting of Lemma 31 the sets  $\omega[u(t)] \cap \mathcal{S}$  and  $\omega[x_n] \cap \mathcal{S}$  contain, at most, one element.

If  $\mathcal{S} \neq \emptyset$  and  $J_1^A$  is a compact operator (maps bounded sets to relatively compact sets) then  $\omega[u(t)] \neq \emptyset$  for every trajectory  $u$  satisfying (5) (see Theorem 11.8 in [54, Pazy]) and  $\omega[x_n] \neq \emptyset$  for every sequence  $\{x_n\}$  satisfying (8).

For instance, if  $A = \partial f$  and the set  $\{u \in H \mid f(u) + \|u\|^2 \leq M\}$  is compact for each  $M \geq 0$ , then  $J_1^A$  is compact. This case was first studied in [19, Brézis].

**Symmetry.** An operator  $A$  is odd if  $D(A) = -D(A)$  and  $A(-x) = -Ax$  for all  $x \in D(A)$ .<sup>3</sup> This is the case, for instance, if  $A = \partial f$  and  $f$  is even. If  $A$  is odd, the semigroup generated is odd as well (see, for instance, [54, Pazy]). On the other hand, it is easy to see that  $J_\lambda^A$  is odd for each  $\lambda > 0$  if  $A$  is odd.

Notice also that if  $A$  is odd then  $\mathcal{S} \neq \emptyset$ . Moreover,  $0 \in \mathcal{S}$ . To see this, take  $x \in D(A)$  and let  $[x, y], [-x, -y] \in A$ . We have

$$\begin{aligned} 4\langle y - 0, x - 0 \rangle &= \langle y + y, x + x \rangle \\ &= \langle y - (-y), x - (-x) \rangle \\ &\geq 0. \end{aligned}$$

Then  $0 \in A0$  by Lemma 1.

**Asymptotic regularity.** A trajectory  $u$  is *asymptotically regular* if  $\lim_{t \rightarrow \infty} \|u(t+h) - u(t)\| = 0$  for each  $h \geq 0$ . A sequence  $\{x_n\}$  is asymptotically regular if  $\lim_{n \rightarrow \infty} \|x_{n+m} - x_n\| = 0$  for each  $m \in \mathbf{N}$ .

*Comments*

Recall that the notion of weak asymptotic regularity was mentioned in Proposition 39 as a characterization of weak convergence of the trajectories satisfying (5).  $\square$

## 7.1 Continuous dynamics

**Strong monotonicity.**

**Proposition 59** *If  $A$  is  $\alpha$ -strongly monotone for some  $\alpha > 0$  then  $u(t)$  converges strongly to the unique  $p \in \mathcal{S}$  as  $t \rightarrow \infty$ .*

**Proof.** Strong monotonicity implies

$$\frac{1}{2} \frac{d}{dt} \|u(t) - p\|^2 = \langle \dot{u}(t), u(t) - p \rangle \leq -\alpha \|u(t) - p\|^2$$

and so  $\|u(t) - p\| \leq e^{-2\alpha t} \|x - p\|$ .  $\blacksquare$

*Comments*

The previous result can be extended in the following way: Let  $X$  be a Banach space such that  $X$  and  $X^*$  are

<sup>3</sup>A weaker notion is that  $A^0(-x) = -A^0x$ . The results below still hold but the proofs become more technical.

uniformly convex. In [50, Nevanlinna and Reich] the authors prove that if  $A$  satisfies NR convergence condition then  $u(t)$  converges strongly to a point in  $\mathcal{S}$  as  $t \rightarrow \infty$ . If only  $X^*$  is uniformly convex, the result remains true provided  $Ax$  is proximal and convex for every  $x$  (see [26, Bruck and Reich]). If neither  $X$  nor  $X^*$  is uniformly convex, the result is still true if the semigroup is differentiable (see [50, Nevanlinna and Reich]).  $\square$

### Solution set with nonempty interior.

**Proposition 60** *Assume  $\text{int } \mathcal{S} \neq \emptyset$ . Then  $u(t)$  converges strongly as  $t \rightarrow \infty$  to a point in  $\mathcal{S}$ .*

**Proof.** If  $B(p, r) \subset \mathcal{S}$ , inequality (36) implies

$$\begin{aligned} r\|u(t) - u(s)\| &\leq r \int_s^t \|\dot{u}(\tau)\| d\tau \\ &\leq - \int_s^t \langle \dot{u}(\tau), u(\tau) - p \rangle d\tau \\ &= \frac{1}{2}\|u(s) - p\|^2 - \frac{1}{2}\|u(t) - p\|^2. \end{aligned}$$

Since  $\|u(t) - p\|$  is convergent by Corollary 7,  $u(t)$  has the Cauchy property.  $\blacksquare$

### Comments

Theorem 4 in [50, Nevanlinna and Reich] shows that this result remains true if  $X$  and  $X^*$  are uniformly convex. In the same paper, the authors give a counterexample in  $\mathcal{C}([0, 1]; \mathbf{R})$ . See also [26, Bruck and Reich].  $\square$

### Strong precompactness.

**Proposition 61** *If  $\omega[u(t)] \cap \mathcal{S} \neq \emptyset$  then  $u(t)$  converges strongly to some  $p \in \mathcal{S}$ .*

**Proof.** If  $p \in \omega[u(t)] \cap \mathcal{S}$  then  $\|u(t) - p\|$  is decreasing and  $\liminf_{t \rightarrow \infty} \|u(t) - p\| = 0$ . Hence  $u(t) \rightarrow p$  as  $t \rightarrow \infty$ .  $\blacksquare$

### Comments

If  $\mathcal{S}$  has nonempty interior then (see [54, Pazy])  $A$  is demipositive and  $\omega[u(t)] \neq \emptyset$  for every trajectory  $u$  satisfying (5). Every strong cluster point is also a weak cluster point, that must lie in  $\mathcal{S}$  by demipositivity. Hence  $\omega[u(t)] \cap \mathcal{S} \neq \emptyset$  and Proposition 60 can also be deduced from Proposition 61.  $\square$

### Symmetry.

**Proposition 62** *If  $A = \partial f$  and  $f \in \Gamma_0(H)$  is even then  $u(t)$  converges strongly as  $t \rightarrow \infty$  to a point in  $\mathcal{S}$ .*

**Proof.** Take  $s > 0$  and define  $\gamma(t) = \|u(t)\|^2 - \|u(s)\|^2 - \frac{1}{2}\|u(t) - u(s)\|^2$ . For  $t \in [0, s]$  one has

$$\dot{\gamma}(t) = \langle \dot{u}(t), u(t) + u(s) \rangle \leq f(-u(s)) - f(u(t)) = f(u(s)) - f(u(t)) \leq 0.$$

Therefore,  $\gamma(t) \geq \gamma(s) = 0$  and so

$$\frac{1}{2}\|u(t) - u(s)\|^2 \leq \|u(t)\|^2 - \|u(s)\|^2.$$

Since  $0 \in \text{Argmin } f$ ,  $\|u(t)\|$  converges as  $t \rightarrow \infty$  so  $u(t)$  has the Cauchy property.  $\blacksquare$

For general  $A$  one has to assume additional hypotheses on the trajectory:

**Proposition 63** *Let  $A$  be odd. If  $u$  is asymptotically regular then  $u(t)$  converges strongly to some  $p \in \mathcal{S}$  as  $t \rightarrow \infty$ .*

**Proof.** Let us use the semigroup notation  $u(t) = S_t x$ . If  $A$  is odd then  $0 \in \mathcal{S}$  and

$$\begin{aligned} \|S_{t+h+s}x + S_{t+s}x\| &= \|S_{t+h+s}x - S_{t+s}(-x)\| \\ &\leq \|S_{t+h}x - S_t(-x)\| \\ &= \|S_{t+h}x + S_t x\| \end{aligned}$$

for each  $h \geq 0$  so that

$$\lim_{t \rightarrow \infty} \|S_t x + S_{t+h}x\| \leq \|S_t x + S_{t+h}x\|. \quad (37)$$

Since  $0 \in \mathcal{S}$  the limit  $d = \lim_{t \rightarrow \infty} \|S_t x\|$  exists. Moreover, the fact that  $\|2S_t x\| \leq \|S_t x + S_{t+h}x\| + \|S_t x - S_{t+h}x\|$  implies

$$2d \leq \lim_{t \rightarrow \infty} \|S_t x + S_{t+h}x\| \leq \|S_t x + S_{t+h}x\|$$

for each  $t, h$  by asymptotic regularity and inequality (37). Finally,

$$\begin{aligned} \|S_{t+h}x - S_t x\|^2 &= 2\|S_t x\|^2 + 2\|S_{t+h}x\|^2 - \|S_{t+h}x + S_t x\|^2 \\ &\leq 4\|S_t x\|^2 - 4d^2 \end{aligned}$$

and so  $\{S_t x\}$  has the Cauchy property. Its limit  $p$  clearly belongs to  $\mathcal{S}$ . ■

### Comments

Without the asymptotic regularity assumption, strong convergence still holds for the averages when  $A$  is odd (see [8, Baillon]). □

## 7.2 Proximal sequences

### Strong monotonicity.

**Proposition 64** *If  $A$  is  $\alpha$ -strongly monotone for some  $\alpha > 0$  then  $x_n$  converges strongly to the unique  $p \in \mathcal{S}$  as  $n \rightarrow \infty$ .*

**Proof.** Strong monotonicity implies

$$\begin{aligned} \alpha \lambda_n \|x_n - p\|^2 &\leq \langle x_{n-1} - x_n, x_n - p \rangle \\ &= \langle x_{n-1} - p, x_n - p \rangle - \|x_n - p\|^2 \\ &\leq \|x_n - p\| (\|x_{n-1} - p\| - \|x_n - p\|) \end{aligned}$$

so that

$$\alpha \sum_{n=1}^{\infty} \lambda_n \|x_n - p\| \leq \|x_0 - p\| < \infty.$$

Since the sequence  $\|x_n - p\|$  is decreasing this implies  $\lim_{n \rightarrow \infty} \|x_n - p\| = 0$ . ■

### Solution set with nonempty interior.

**Proposition 65** *Let  $A$  be maximal monotone with  $\text{int } \mathcal{S} \neq \emptyset$ . Then  $x_n$  converges strongly as  $n \rightarrow \infty$ .*

**Proof.** If  $B(p, r) \subset \mathcal{S}$  inequality (36) gives  $r\|x_{k-1} - x_k\| \leq \langle x_{k-1} - x_k, x_k - p \rangle$  and so

$$\begin{aligned} r\|x_{k-1} - x_k\| &\leq \langle x_{k-1} - p, x_k - p \rangle - \|x_k - p\|^2 \\ &\leq \|x_0 - p\| (\|x_{k-1} - p\| - \|x_k - p\|) \end{aligned}$$



by Corollary 16. Hence

$$\begin{aligned} r \|x_n - x_m\| &\leq r \sum_{k=n+1}^m \|x_{k-1} - x_k\| \\ &\leq \|x_0 - p\| \left( \|x_n - p\| - \|x_m - p\| \right). \end{aligned}$$

Since  $\|x_n - p\|$  is convergent,  $x_n$  is a Cauchy sequence. ■

### The NR convergence condition.

A fairly general result is the following, from [50, Nevanlinna and Reich]:

**Theorem 66** *If  $A$  satisfies the NR convergence condition then  $x_n$  converges strongly as  $n \rightarrow \infty$ .*

**Proof.** Setting  $j_n = x_n - P_{\mathcal{S}}x_n$  we have

$$\begin{aligned} \|j_n\|^2 + \lambda_n \langle y_n, j_n \rangle &= \langle x_{n-1} - P_{\mathcal{S}}x_n, j_n \rangle \\ &= \langle j_{n-1}, j_n \rangle + \langle P_{\mathcal{S}}x_{n-1} - P_{\mathcal{S}}x_n, x_n - P_{\mathcal{S}}x_n \rangle \\ &\leq \|j_{n-1}\| \|j_n\| \\ &\leq \frac{1}{2} [\|j_{n-1}\|^2 + \|j_n\|^2]. \end{aligned}$$

Thus  $\|j_n\|^2 + 2\lambda_n \langle y_n, j_n \rangle \leq \|j_{n-1}\|^2$  and  $\sum_{n=1}^{\infty} \lambda_n \langle y_n, j_n \rangle < \infty$ . Since  $\langle y_n, j_n \rangle \geq 0$  one must have  $\liminf_{n \rightarrow \infty} \langle y_n, j_n \rangle = 0$ . The sequences  $\{x_n\}$  and  $\{y_n\}$  are bounded, and the convergence condition implies  $\liminf_{n \rightarrow \infty} \|x_n - P_{\mathcal{S}}x_n\| = 0$ . Since  $\|x_n - P_{\mathcal{S}}x_n\|$  is nonincreasing, it must converge to 0. On the other hand, the sequence  $\|x_n - p\|$  is nonincreasing for each  $p \in \mathcal{S}$ . In particular,  $\|x_{n+m} - P_{\mathcal{S}}x_n\| \leq \|x_n - P_{\mathcal{S}}x_n\|$  and therefore  $\|x_{n+m} - x_n\| \leq 2\|x_n - P_{\mathcal{S}}x_n\|$ . We conclude that  $x_n$  converges strongly to some  $p \in \mathcal{S}$  as  $n \rightarrow \infty$ . ■

### Strong precompactness.

**Proposition 67** *If  $\omega[x_n] \cap \mathcal{S} \neq \emptyset$  then  $x_n$  converges strongly to some  $p \in \mathcal{S}$ .*

**Proof.** If  $p \in \omega[x_n] \cap \mathcal{S}$  then  $\|x_n - p\|$  is decreasing and  $\liminf_{n \rightarrow \infty} \|x_n - p\| = 0$ . ■

### Symmetry.

For even functions we have the following result from [20, Brézis and Lions]:

**Proposition 68** *If  $A$  is the subdifferential of an even function in  $f \in \Gamma_0(H)$  then  $x_n$  converges strongly as  $n \rightarrow \infty$ .*

**Proof.** Recall that  $2\lambda_n(f(u) - f(x_n)) \geq \|u - x_n\|^2 - \|u - x_{n-1}\|^2$ . Let  $m \geq n$  and take  $u = -x_m$ . Since  $n \mapsto f(x_n)$  is decreasing we have  $\|x_m + x_n\| \leq \|x_m + x_{n-1}\|$  and the function  $n \mapsto \|x_m + x_n\|$  is decreasing. In particular  $\|x_m + x_m\| \leq \|x_m + x_n\|$ , thus  $4\|x_m\|^2 \leq \|x_m + x_n\|^2$ . We have  $2\|x_n\|^2 + 2\|x_m\|^2 = \|x_m + x_n\|^2 + \|x_m - x_n\|^2 \geq 4\|x_m\|^2 + \|x_m - x_n\|^2$ , so that  $\|x_m - x_n\|^2 \leq 2\|x_n\|^2 - 2\|x_m\|^2$ . Since  $\|x_n\|$  converges as  $n \rightarrow \infty$  this proves that  $x_n$  is a Cauchy sequence. ■

As before, asymptotic regularity is required for a general  $A$ :

**Proposition 69** *Let  $A$  be odd. If  $\{x_n\}$  is asymptotically regular then  $x_n$  converges strongly to some  $p \in \mathcal{S}$  as  $n \rightarrow \infty$ .*

**Proof.** We already proved that  $0 \in \mathcal{S}$  when  $A$  is odd. Next, one easily verifies that the sequence  $\|x_{n+k} + x_n\|$  is decreasing for each  $k \in \mathbf{N}$  and concludes as in the proof of Proposition 63. ■

### Comments

Without asymptotic regularity one can still prove strong convergence of the averages (see [43, Lions]) for odd operators. This was first proved in [9, Baillon] in the case  $\lambda_n \equiv \lambda$ . □

### 7.3 Euler sequences

#### Strong monotonicity.

**Proposition 70** *Let  $A$  be  $\alpha$ -strongly monotone. If  $\sum \|z_n - z_{n-1}\|^2 < \infty$  then  $z_n$  converges strongly to the unique  $p \in \mathcal{S}$  as  $n \rightarrow \infty$ .*

**Proof.** The strong monotonicity and (21) together imply

$$2\alpha\lambda_n\|z_n - p\|^2 + \|z_{n+1} - p\|^2 \leq \|z_n - p\|^2 + \lambda_n^2\|w_n\|^2.$$

Therefore

$$2\alpha \sum_{n=1}^{\infty} \lambda_n \|z_n - p\|^2 \leq \|z_0 - p\|^2 + \sum \lambda_n^2 \|w_n\|^2 < \infty$$

and so  $\liminf_{n \rightarrow \infty} \|z_n - p\| = 0$ . But  $\|z_n - p\|$  converges by Corollary 23. ■

#### Solution set with nonempty interior.

**Proposition 71** *Assume  $\text{int } \mathcal{S} \neq \emptyset$ . If  $\sum \|z_n - z_{n-1}\|^2 < \infty$  then  $z_n$  converges strongly as  $n \rightarrow \infty$ .*

**Proof.** If  $B(p, r) \subset \mathcal{S}$  inequalities (36) and (22) together give

$$2r\lambda_n\|w_n\| + \|z_{n+1} - p\|^2 \leq \|z_n - p\|^2 + \lambda_n^2\|w_n\|^2.$$

This implies that the sequence  $\lambda_n\|w_n\| = \|z_{n+1} - z_n\|$  is in  $\ell^1$  and so  $z_n$  converges. ■

#### The NR convergence condition.

**Theorem 72** *Assume  $\sum \|z_{n+1} - z_n\|^2 < \infty$  and  $w_n$  is bounded. If  $A$  satisfies the NR convergence condition then  $\{z_n\}$  converges strongly as  $n \rightarrow \infty$ .*

**Proof.** To simplify notation write  $j_n = z_n - P_S z_n$ . We have

$$\|j_{n+1}\|^2 \leq \|z_{n+1} - P_S z_n\|^2 = \|j_n + \lambda_n w_n\|^2 = \|j_n\|^2 - 2\lambda_n \langle -w_n, j_n \rangle + \lambda_n^2 \|w_n\|^2.$$

By hypothesis and Corollary 23 the sequence  $\{z_n, -w_n\}$  is bounded. Moreover,

$$\sum_{n=1}^{\infty} \lambda_n \langle -w_n, j_n \rangle < \infty.$$

But  $\langle -w_n, j_n \rangle \geq 0$  and so  $\liminf_{n \rightarrow \infty} \langle w_n, j_n \rangle = 0$  and the convergence condition implies  $\liminf_{n \rightarrow \infty} \|j_n\| = 0$ . This sequence being convergent we have  $\lim_{n \rightarrow \infty} j_n = 0$ . Finally,  $\|z_{n+m} - z_n\| \leq 2\|j_n\|$  and so  $z_n$  converges as  $n \rightarrow \infty$ . ■

#### Comments

The previous result holds if  $X$  and  $X^*$  are uniformly convex (see [50, Nevanlinna and Reich]).

According to [26, Bruck and Reich], the convergence condition can be replaced by  $\text{int } \mathcal{S} \neq \emptyset$ . In that case, if  $X$  is not uniformly convex it suffices that  $Ax$  be proximal and convex for each  $x$ . On the other hand, according to [50, Nevanlinna and Reich], the conclusion of Theorem 72 is still true, even if  $X$  and  $X^*$  are not uniformly convex, provided  $\mathcal{S}$  is proximal and  $A$  is *accretive in the sense of Browder*. □

#### Strong precompactness.

**Proposition 73** *Assume that  $\sum \|z_{n+1} - z_n\|^2 < \infty$  and  $\omega[z_n] \cap \mathcal{S} \neq \emptyset$ . Then  $z_n$  converges strongly to some  $p \in \mathcal{S}$ .*

**Proof.** The argument is the same as in Proposition 61 by virtue of Corollary 23. ■

### Symmetry.

The following results uses the same ideas as in Propositions 63 and 69 but is apparently new:

**Proposition 74** *Let  $T$  be nonexpansive,  $A = I - T$  and  $\lambda_n \equiv 1$  so that  $z_n = T^n z_0$ . If  $T$  is odd and  $\sum \|z_{n+1} - z_n\|^2 < \infty$  then  $\{z_n\}$  is strongly convergent.*

**Proof.** Since  $T$  is odd one easily deduces that the sequence  $\|z_{n+k} + z_n\|$  is decreasing for each  $k$ . From the fact that  $\sum \|z_{n+1} - z_n\|^2 < \infty$  we can draw two conclusions: In the first place, Corollary 23 implies  $d = \lim_{n \rightarrow \infty} \|z_n\|$  exists because  $0 \in \mathcal{S}$ . On the other hand, the sequence  $z_n$  is asymptotically regular, so  $\lim_{n \rightarrow \infty} \|z_n - z_{n+k}\|$  exists for each  $k$ . As a consequence,  $2d \leq \|z_{n+k} + z_n\|$  for each  $n$  and  $k$ . One concludes as in the proof of Proposition 63. ■

### Comments

Without any further assumptions, the sequence  $T^n z$  converges strongly in average if  $T$  is odd (see [9, Baillon]). □

## 8 Asymptotic equivalence

In this section we explain how to deduce qualitative information on the asymptotic behavior of the systems defined by (5), (8) and (17). We provide a comparison tool that guarantees that two evolution systems share certain asymptotic properties. For the complete abstract theory see [3, Alvarez and Peypouquet].

### 8.1 Evolution systems

Let  $C$  be a convex subset of a Banach space  $X$  and let  $I$  denote the identity operator in  $X$ . An *evolution system* (ES) on  $C$  is a family  $\{V(t, s) : t \geq s \geq 0\}$  of maps from  $C$  into itself satisfying:

- i)  $V(t, t) = I$ ; and
- ii)  $V(t, s)V(s, r) = V(t, r)$ .

Let  $L > 0$ . An evolution system is *L-Lipschitz* if it satisfies

- iii)  $\|V(t, s)x - V(t, s)y\| \leq L\|x - y\|$

and is *contracting* (CES) if it is 1-Lipschitz.

**Example 5** Let  $F$  be a (possibly multivalued) function from  $[t_0, \infty) \times C$  to  $C$ . Suppose that for every  $s \geq t_0$  and  $x \in C$  the differential inclusion  $u'(t) \in F(t, u(t))$ , with initial condition  $u(s) = x$ , has a unique solution  $u_{s,x} : [s, \infty) \mapsto C$ . The family  $U$  defined by  $U(t, s)x = u_{s,x}(t)$  is an evolution system on  $C$ . If  $X$  is Hilbert space and  $F(t, x) = -A_t x$ , where  $\{A_t\}$  is a family of maximal monotone operators, then the corresponding  $U$  is a CES. □

**Example 6** Take a strictly increasing unbounded sequence  $\{\sigma_n\}$  of positive numbers and set  $\nu(t) = \max\{n \in \mathbf{N} \mid \sigma_n \leq t\}$ . Consider a family  $\{F_n\}$  of functions from  $C$  into  $C$  and define  $U(t, s) = \prod_{n=\nu(s)+1}^{\nu(t)} F_n$ , the product representing composition of functions. Then  $U$  is an ES. If each  $F_n$  is  $M_n$ -Lipschitz and the product  $\prod_{n=1}^{\infty} M_n$  is bounded from above by  $M$ , then  $U$  is an  $M$ -LES. For instance, if  $F_n = (I + A_n)^{-1}$ , where  $\{A_n\}$  is a family of  $m$ -accretive operators on  $C$ , then the piecewise constant interpolation of infinite products of resolvents defines a CES. □

## 8.2 Almost-orbits and asymptotic equivalence

Let  $V$  be an evolution system on  $C$ . A locally bounded trajectory of the form  $t \mapsto V(t, s)x$  for  $s$  and  $x$  fixed is an orbit of  $V$ . A locally bounded function  $u : \mathbf{R}_+ \rightarrow C$  is an *almost-orbit* of  $V$  if

$$\lim_{t \rightarrow \infty} \|u(t+h) - V(t+h, t)u(t)\| = 0 \quad \text{uniformly in } h \geq 0. \quad (38)$$

Orbits and almost-orbits have, essentially, the same asymptotic behavior.

Note the relation and difference with the notion of asymptotic pseudotrajectories where the convergence is uniform on compact time intervals ([15, Benaïm and Hirsch], [16, Benaïm, Hofbauer and Sorin]). The current concept is more demanding but will allow for more precise results (convergence rather than properties on the set of limit points). The following result is from [3, Alvarez and Peypouquet] (see also [58, Peypouquet]):

**Theorem 75** *Let  $V$  be an evolution system. For the weak topology assume either that  $V$  is Lipschitz or  $X$  is weakly complete (weak Cauchy nets are weakly convergent<sup>4</sup>). If every orbit of  $V$  converges weakly (resp. strongly), then so does every almost-orbit.*

**Proof.** For the strong topology, let  $u$  be an almost-orbit of  $V$  and let  $\varepsilon > 0$ . By definition, there is  $S > 0$  such that

$$\|u(t+h) - V(t+h, t)u(t)\| < \varepsilon/4$$

for all  $h \geq 0$  and  $t \geq S$ . Define  $\zeta(S) = \lim_{t \rightarrow \infty} V(t, S)u(S)$  and choose  $T > S$  such that  $\|V(t, S)u(S) - \zeta(S)\| < \varepsilon/4$  for all  $t \geq T$ . Then

$$\|u(t+h) - \zeta(S)\| \leq \|u(t+h) - V(t+h, S)u(S)\| + \|V(t+h, S)u(S) - \zeta(S)\| < \varepsilon/2$$

for all  $t \geq T$  and all  $h \geq 0$ . Thus  $\|u(t') - u(t)\| < \varepsilon$  for all  $t, t' \geq T$  so that  $u(t)$  is Cauchy and converges. It is clear that this argument is valid for the weak topology if  $X$  is weakly complete.

If it is not the case but  $V$  is  $L$ -Lipschitz, one defines  $\zeta(s) = \tau - \lim_{t \rightarrow \infty} V(t, s)u(s)$  (where  $\tau$  is weak or strong) and verifies that

$$\sup_{p \geq 0} \|\zeta(s+p) - \zeta(s)\| \leq L \sup_{p \geq 0} \|u(s+p) - V(s+p, s)u(s)\|,$$

which tends to zero as  $s \rightarrow \infty$  showing that  $\zeta(s)$  converges strongly to some  $\zeta$ . Decompose now, for  $t \geq s$

$$u(t) - \zeta = [u(t) - V(t, s)u(s)] + [V(t, s)u(s) - \zeta(s)] + [\zeta(s) - \zeta].$$

The last two terms converge to 0 in norm as  $s \rightarrow \infty$  (uniformly in  $t$ ) and the first one  $\tau$ -converges to 0 as  $t \rightarrow \infty$  for each  $s$ . Hence  $u(t)$   $\tau$  converges to  $\zeta$  as  $t \rightarrow \infty$ . ■

A special case of Theorem 75 was proved in [53, Passty], when  $V$  is defined by a semigroup of contractions or if the almost-orbits are orbits of a semigroup of contractions.

**Theorem 76** *Under the hypotheses of Theorem 75, the conclusion remains valid if the word converges is replaced by converges in average.*

The proof of this result can be found in [3, Alvarez and Peypouquet].

### Comments

Under the hypotheses of Theorem 75, the conclusion is also true for *almost-convergence* (see [3, Alvarez and Peypouquet]), a concept developed in [44, Lorentz] that is stronger than convergence in average. This result had been proved earlier in [48, Miyadera and Kobayasi] under supplementary assumptions: i)  $V$  is defined by a strongly continuous semigroup of contractions; ii)  $\mathcal{S} \neq \emptyset$ ; and iii) for the weak topology,  $X$  is weakly complete. □

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<sup>4</sup>The spaces  $\ell^1$  and  $L^1$ , as well as all reflexive Banach spaces, have this property. It is not the case if  $X$  contains  $c_0$ , though (see p. 88 in [42, Li and Queffélec]).

**Lemma 77** *Let  $U$  and  $V$  be evolution systems and assume that for each  $r > 0$*

$$\limsup_{t \rightarrow \infty} \sup_{h \geq 0} \sup_{\|z\| \leq r} \|U(t+h, t)z - V(t+h, t)z\| = 0$$

*then every bounded orbit of  $V$  is an almost-orbit of  $U$  and viceversa.*

**Proof.** Let  $v$  be an orbit of  $V$  such that  $\|v(t)\| \leq r$  for all  $t$ . Then

$$\begin{aligned} \|v(t+h) - U(t+h, t)v(t)\| &= \|V(t+h, t)v(t) - U(t+h, t)v(t)\| \\ &\leq \sup_{\|z\| \leq r} \|U(t+h, t)z - V(t+h, t)z\| \end{aligned}$$

and so  $v$  is an almost-orbit of  $U$ . ■

### 8.3 Continuous dynamics and discretizations

The following results explain why in most cases the systems defined in the preceding sections converge under the same hypotheses. The proofs are considerably simplified if one assumes boundedness of the almost-orbits by virtue of Lemma 77. We provide the proofs in this case. For the general setting the reader can consult the original reference. The next proposition gathers results from [65, Sugimoto and Koizumi] and [33, Güler].

**Proposition 78** *Let  $A$  be a maximal monotone operator on  $H$  and let  $U$  and  $V$  be the evolution systems defined by the differential inclusion (5) and the proximal algorithm (8), respectively. Assume one of the following conditions holds:*

- i)  $\{\lambda_n\} \in \ell^2 \setminus \ell^1$ ; or*
- ii)  $A = \partial f$  and  $\{\lambda_n\} \notin \ell^1$ .*

*Then every orbit of  $U$  is an almost-orbit of  $V$  and viceversa.*

**Proof.** Define  $\nu(t)$  as in Example 6. If  $\{\lambda_n\} \in \ell^2 \setminus \ell^1$ , part *i*) in Corollary 20 gives

$$\|U(t+s, t)z - V(t+s, t)z\|^2 \leq 3\|A^0 z\|^2 \sum_{n=\nu(t)}^{\infty} \lambda_n^2$$

and we conclude using Lemma 77. For unbounded almost-orbits, see [65, Sugimoto and Koizumi]. If  $A = \partial f$  and  $\{\lambda_n\} \notin \ell^1$  the proof is highly technical and can be found in [33, Güler]. It also relies on part *i*) in Corollary 20 but sharper estimations on  $\|A^0 x_n\|$  and  $\|A^0 u(t)\|$  are needed. ■

The following result is from [4, Alvarez and Peypouquet] (see also [58, Peypouquet]):

**Proposition 79** *Let  $T$  be nonexpansive, set  $A = I - T$  and let  $U$  and  $W$  be the evolution systems defined by the differential inclusion (5) and Euler's discretization (17), respectively. Assume  $\{\lambda_n\} \in \ell^2 \setminus \ell^1$ . Then every orbit of  $U$  is an almost-orbit of  $W$  and viceversa.*

**Proof.** The argument in the proof of part *i*) in Proposition 78 can be applied here as well, by virtue of inequality (18). ■

These properties allow for a better understanding of similar asymptotic behavior of the continuous and discrete processes: in general for weak convergence in average (Section 4), for weak convergence in the case of demi-positive operators (Section 5) and for strong convergence under additional geometrical hypotheses (Section 6).

### 8.4 Quasi-autonomous systems

One of the advantages of this approach through almost-orbits is that it extends to nonautonomous systems, which arise naturally in the presence of perturbations.

### 8.4.1 Continuous dynamics

Recall that the solutions of the differential inclusion (5) define an evolution system  $U$  as in Example 5. Let us consider quasi-autonomous versions of (5), namely

$$-\dot{v}(t) \in Av(t) + \varphi(t) \quad (39)$$

and

$$-\dot{v}(t) \in Av(t) + \varepsilon(t)v(t). \quad (40)$$

The following result is from [4, Alvarez and Peypouquet] (see also [58, Peypouquet]):

**Proposition 80** *i) If  $\varphi \in L^1(0, \infty; X)$ , then every function  $v$  satisfying (39) is an almost-orbit of  $U$ .  
ii) The same holds for every function satisfying (40) provided  $\varepsilon \in L^1(0, \infty; \mathbf{R})$ .*

**Proof.** For the first part we follow [48, Miyadera and Kobayasi]. If  $v$  satisfies (39) and  $t \geq 0$  we have, with  $V(t+s) = U(t+s, t)v(t)$

$$\begin{aligned} \|v(t+s) - U(t+s, t)v(t)\|^2 &= \|v(t+s) - V(t+s)\|^2 \\ &= 2 \int_0^s \langle \dot{v}(t+\tau) - \dot{V}(t+\tau), v(t+\tau) - V(t+\tau) \rangle d\tau \\ &\leq 2 \int_0^s \langle -\varphi(t+\tau), v(t+\tau) - V(t+\tau) \rangle d\tau \\ &\leq 2 \int_0^s \|\varphi(t+\tau)\| \|v(t+\tau) - V(t+\tau)\| d\tau \\ &= 2 \int_0^s \|\varphi(t+\tau)\| \|v(t+\tau) - U(t+\tau, t)v(t)\| d\tau \end{aligned}$$

and so

$$\|v(t+s) - U(t+s, t)v(t)\| \leq \int_0^s \|\varphi(t+\tau)\| d\tau \leq \int_t^\infty \|\varphi(\tau)\| d\tau.$$

For *ii)*, let  $v$  satisfy (40). Fix  $t$  and consider as above  $\psi(s) = \frac{1}{2}\|U(t+s, t)v(t) - v(t+s)\|^2 = \frac{1}{2}\|V(t+s) - v(t+s)\|^2$ . Hence for almost every  $s > 0$ ,

$$\dot{\psi}(s) \leq \varepsilon(t+s) \langle v(t+s), V(t+s) - v(t+s) \rangle \leq \frac{1}{4} |\varepsilon(t+s)| \|U(t+s, t)v(t)\|^2$$

(using  $\langle \zeta, \xi - \zeta \rangle \leq \frac{1}{4}\|\xi\|^2$  for all  $\zeta, \xi \in H$ ). Integrating from 0 to  $s$  and observing that  $\psi(0) = 0$  we obtain

$$\|U(t+s, t)v(t) - v(t+s)\|^2 \leq \frac{1}{4} \int_0^s |\varepsilon(t+\tau)| \|U(t+\tau, t)v(t)\|^2 d\tau \leq \frac{M}{4} \int_t^\infty |\varepsilon(\tau)| d\tau$$

if  $v$  is bounded. ■

#### Comments

In [1, Alvarez], the author studies the problem

$$u''(t) + \gamma u'(t) + \nabla \Phi(u(t)) = 0, \quad (41)$$

where  $\Phi$  is a  $C^1$  convex function. He proves that if  $\text{Argmin}(\Phi) \neq \emptyset$ , then each solution  $u(t)$  converges weakly to a minimizer of  $\Phi$  as  $t \rightarrow \infty$  and gives conditions for strong convergence. Later, in [6, Attouch and Czarnecki] the authors establish, among other results, that if  $\varepsilon \in L^1$  the solutions of

$$u''(t) + \gamma u'(t) + \nabla \Phi(u(t)) + \varepsilon(t)u(t) = 0. \quad (42)$$

also converge weakly to minimizers of  $\Phi$ . It turns out (see [4, Alvarez and Peypouquet] or [58, Peypouquet]) that under this condition ( $\varepsilon \in L^1$ ) the solutions of (42) are almost-orbits of the evolution system defined by (41).

This is an alternative way to prove the cited result from [6, Attouch and Czarnecki] and it shows that these tools building on almost-orbits to classify the asymptotic behavior through equivalence classes (continuous trajectories, proximal or Euler approximations, Tykhonov regularization, perturbations) can be applied to second-order systems as well. Moreover, if  $u''$  is integrable, the results for quasi-autonomous systems imply that the orbits of (41) are almost-orbits of the corresponding first-order evolution system. □

### 8.4.2 Proximal sequences

In a similar fashion one can prove any interpolation of a sequence  $\{y_n\}$  satisfying

$$y_{n-1} - y_n \in \lambda_n A y_n + \phi_n \quad (43)$$

or

$$y_{n-1} - y_n \in \lambda_n A y_n + \epsilon_n y_n \quad (44)$$

is an almost-orbit of the evolution system  $U$  defined by the proximal scheme (8) as in Example 6 provided  $\{\phi_n\} \in \ell^1(\mathbf{N}; X)$  and  $\{\epsilon_n\} \in \ell^1(\mathbf{N}; \mathbf{R}_+)$ , respectively.

For additional applications and examples see [4, Alvarez and Peypouquet] (or [58, Peypouquet]).

## 9 Concluding remarks

It is useful to observe that there are two aspects related to the ideas of asymptotic equivalence discussed in the last section. In the first place, one can obtain sufficient conditions for a perturbed, regularized or discretized system to have the same asymptotic properties as the original one. The issue here is in terms of stability, regularity or computational purposes. On the other hand, if a given dynamics does not have some desirable asymptotic behavior, one can introduce perturbation in order to generate orbits having better properties. In this case, the tools of asymptotic equivalence give necessary conditions for a perturbation to be effective.

Recall that the trajectories defined by (5) only converge weakly in average. Even in the case where  $A = \partial f$ , convergence is still weak and the limit depends on the initial point. One can get a better asymptotic behavior by forcing the system to stabilize, for instance, in the direction of the origin. More precisely, consider a piecewise absolutely continuous function  $\varepsilon : \mathbf{R}_+ \rightarrow \mathbf{R}_+$  such that  $\lim_{t \rightarrow \infty} \varepsilon(t) = 0$ . If  $\varepsilon \in L^1(0, \infty; \mathbf{R}_+)$  the system defined by (40) will have the same asymptotic behavior as (5) by Proposition 80. If we expect the regularized system to have better properties we must consider  $\varepsilon \notin L^1(0, \infty; \mathbf{R}_+)$ . The following result is from [28, Cominetti, Peypouquet and Sorin]:

**Proposition 81** *Suppose  $v : \mathbf{R}_+ \rightarrow H$  satisfies*

$$-\dot{v}(t) \in Av(t) + \varepsilon(t)v(t).$$

*with  $\varepsilon \notin L^1(0, \infty; \mathbf{R}_+)$ . Assume further that  $A = \partial f$  or  $\int_0^\infty |\dot{\varepsilon}(t)| dt < \infty$  (finite total variation). Then  $\lim_{t \rightarrow \infty} v(t) = P_S 0$ .*

Special cases of the preceding result had been proved earlier in [21, Browder], [59, Reich] and [5, Attouch and Cominetti]. A similar result for the second order appears in [6, Attouch and Czarnecki].

Similarly is one applies the proximal point algorithm with step sizes  $\lambda_n \in \ell^2$  to the continuous dynamics (5), by Proposition 78 the corresponding system will have the same asymptotic properties, hence need not be weakly convergent. In other words, the approximation is too good: “the discrete approximation mirrors the behavior of the differential equation *too well*” [25, Bruck, p. 29]. If one wishes to get a better (or different) behavior, it is necessary to consider  $\lambda_n \notin \ell^2$ . This turns out to be fruitful because, in that case Theorem 53 guarantees weak convergence even when the operator is not demipositive (see also Example 3 in Section 6).

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